THE THEORY AND PRACTICE OF SOLUTION FOR FUZZY RELATIVE EQUATIONS IN MAX-PRODUCT

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This paper studies the existence in solution of Fuzzy relative equations in Max-product and the theorem for maximum solution and minimum solution; Then it gives a short-circuit for solution. And at the same time, with this relative equations, it solves the influence factor for economical benefits in interprises of commerce, the result of which tallies with practice basically.

Keywords: Max-product Fuzzy relative equations (M-PFRE), Finite field, Influence factor, Maximum solution, Minimum solution, Matrix of candidate solution.

1. Introduction

Let $U=\{x_1, x_2, \cdots, x_p\}$, $V=\{Y_1, Y_2, \cdots, Y_q\}$ $(P\geqslant m,\ q\geqslant n)$ be finite field, and Fuzzy relation $\underset{\sim}{A}\in F(U\times V)$, $\underset{\sim}{X}\in F(U)$, $\underset{\sim}{B}\in F(V)$, then consider the generalized Fuzzy relative equations:

$$\underset{\sim}{A}\circ\underset{\sim}{X}=\underset{\sim}{B} \tag{1.1}$$

where "o" represents Max-product operation, that is operator $(\vee,\cdot)$.

Let $\underset{\sim}{B}=(b_1, b_2, \cdots b_n)^T$ and

$$\underset{\sim}{A}=\begin{pmatrix} \underset{\sim}{A}_1(x_1^{(1)}) & \underset{\sim}{A}_2(x_2^{(1)}) & \cdots & \underset{\sim}{A}_m(X_m^{(1)}) \\ \cdots & \cdots & \cdots & \cdots \\ \underset{\sim}{A}_1(x_1^{(n)}) & \underset{\sim}{A}_2(x_2^{(n)}) & \cdots & \underset{\sim}{A}_m(X_m^{(n)}) \end{pmatrix} \tag{1.2}$$

how to solve $\underset{\sim}{X}=(x_1, x_2, \cdots x_m)^T$? "$T$" represents transpose.

First of all, this paper studies the solution of (1.1) theoretically, then, by the application of practical examples, attains the degree factor influencing economical benefits in enterprises of commerce, which finds the practical background for the application of equations (1.1).

II. The Solubility of the M-PFRE and the Theorem for the Maximum Solution

In order to discuss conveniently, let matrix element in (1·2) be:

$$\underset{\sim}{A}_j(x_j^{(i)}) = a_{ij}. \quad (i=1,2,\cdots,n;\ j=1,2,\cdots,m)$$

Then, next step, we only discuss Fuzzy relative equations like this:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ & \ddots & & \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{pmatrix} \circ \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \qquad (2\cdot1)$$

where the compound operation "o" in matrix be Max-product composition, i·e $\bigvee_{1\le j\le m}(a_{ij}\cdot x_j)=b_i \quad (i=1,2,\cdots,n)$

Definition 2·1
$$a_{ij}\,\alpha^{-1}\,b_i \triangleq \begin{cases} \dfrac{b_i}{a_{ij}} & a_{ij}>b_i \\ 1 & a_{ij}\le b_i \end{cases} \qquad \forall a_{ij}, b_i\in[0,1] \qquad (2\cdot2)$$

where α^{-1} is an operator defined at $[0,1]$. And let

$$K_j \triangleq \bigwedge_{i=1}^{n} a_{ij}\ \alpha^{-1} b_i \quad (j=1,2,\cdots,m) \qquad (2\cdot3)$$

then $\underset{\sim}{k}=(K_1,K_2,\cdots,K_m)\in\underset{\sim}{X}$ is the maximum element in $\underset{\sim}{X}$.

Proposition 2·1 $\quad b\geqslant c \Rightarrow a\alpha^{-1}b \geqslant a\alpha^{-1}c$

Proof. $\quad a>b\Rightarrow a>c$, From (2·2), $a\alpha^{-1}b=\frac{b}{a}\geqslant\frac{c}{a}=a\alpha^{-1}c$.

$a\leqslant b\Rightarrow a\alpha^{-1}b=1$, but $a\alpha^{-1}c\leqslant 1$, hence $a\alpha^{-1}b\geqslant a\alpha^{-1}c$.

Inference 2·1. $a\alpha^{-1}(b\vee c)\geqslant a\alpha^{-1}c$.

Proposition 2·2. $a\cdot(a\alpha^{-1}b)=a\wedge b;\ a\alpha^{-1}(a\cdot b)\geqslant b$.

Proof: 1° $\quad a>b\Rightarrow a\alpha^{-1}b=\frac{b}{a}\Rightarrow a\cdot(a\alpha^{-1}b)=b$.

$a\leqslant b\Rightarrow a\alpha^{-1}b=1\Rightarrow a\cdot(a\alpha^{-1}b)=a\cdot 1=a$.

So $\quad a\cdot(a\alpha^{-1}b)=a\wedge b$.

2° When $a>ab\Rightarrow a\alpha^{-1}(a\cdot b)=b$; $a\leqslant ab\Rightarrow a\alpha^{-1}(a\cdot b)=1$

So $\quad a\alpha^{-1}(a\cdot b)\geqslant b$

Theorem 2·1 There exists solution $\underset{\sim}{X}=(x_1,x_2,\cdots,x_m)^T$ in Fuzzy relative equations (2·1) if and only if

$a_{ij}x_j \leqslant b_i$ $(i\leqslant n, j\leqslant m$, and for each j there exists j_i such that $a_{ij_i}=b_i$.

Proof: Sufficiency shows. Now let's prove the necessity.

If $X=(x_1,x_2,\cdots x_m)^T$ is the solution for (2·1), then

$$a_{ij}\cdot x_i \leqslant b_i \quad (i\leqslant n, j\leqslant m) \tag{2·4}$$

otherwise, if there exists i,j, such that $a_{ij}x_j > b_i$, then

$$(a_{i1}\cdot x_1)\vee\cdots\vee(a_{ij}\cdot x_j)\vee\cdots\vee(a_{im}\cdot x_m) > b_i$$

Contradictory! So (2·4) holds.

At the same time, if there exists solution in (2·4), and $a_{ij}\cdot x_j\leqslant b_i$ $(i\leqslant n,\ j\leqslant m)$, there must exist j_i for each $i\leqslant n$ so that $a_{ij_i}x_{ji}=b_i$ $(i\leqslant n)$ otherwise.

1° We have proved it impossible that if for each i there exists j_i such that $a_{ij_i}x_{ji}>b_i$.

2° If for each j there exists i such that $a_{ij}x_j<b_i$, then

$$(a_{i1}x_1)\vee(a_{i2}x_2)\vee\cdots\vee(a_{im}x_m)<b_i$$

which is in contradictory with the solution in (2·1).

In practical application, (2·1) probably has no solution, but small alteration may be always given to $\underset{\sim}{A}$ for ε and $\underset{\sim}{B}$ for δ, so that

$$\underset{\sim}{A}^{\varepsilon}\circ\underset{\sim}{X}=\underset{\sim}{B} \qquad \text{or} \qquad \underset{\sim}{A}\circ\underset{\sim}{X}=\underset{\sim}{B}^{\delta}$$

has an steady solution.

So the following is always assumed to have solution in (2·1). If $\underset{\sim}{B}$ in (2·1) is arranged in standardization, then

$$b_1'\geqslant b_2'\geqslant\cdots\geqslant b_n' \quad (\text{or } b_1'\leqslant b_2'\leqslant\cdots\leqslant b_n')$$

For shortly. let b_i still stand for b_i' and (a_{ij}) for (a_{ij}') correspondingly.

Theorem 2·2. If there exists solution $R\neq\phi$ in (2·1), then $\underset{\sim}{K}$ is the maximum solution of it.

proof: Because $R\neq\phi$, then $\{i, a_{ij}>b_i\}\neq\phi$ $(i\leqslant n, j\leqslant m)$. Hence when $\underset{\sim}{A}\circ\underset{\sim}{K}=(b_i';\ i\leqslant n)$, then

$$b_i'=\bigvee_{i=1}^{n}[a_{ij}\cdot(\bigwedge_{i=1}^{n}(a_{ij}\alpha^{-1}b_i))]=\bigvee_{i=1}^{n}[a_{ij}\cdot(a_{ij}\alpha^{-1}b_i)]$$

$$\bigvee_{i=1}^{n}(a_{ij}\cdot\frac{b_i}{a_{ij}})=b_i.\quad(i\leqslant n).$$

Again $\underset{\sim}{X}$, $\underset{\sim}{K} \in R$, then:

$$k_{j0} = \bigwedge_{i=1}^{n} (a_{ij_0} \alpha^{-1} b'_i) = \bigwedge_{i=1}^{n} (a_{ii_0} \alpha^{-1} b_i)$$

$$= \bigwedge_{i=1}^{n} [a_{ij_0} \alpha^{-1} (\bigvee_{i=1}^{n} a_{ij_0} \cdot x_{j0})] \geqslant \bigwedge_{i=1}^{n} a_{ij_0} \alpha^{-1} (a_{ij_0} \cdot x_{j0})]$$

$$= a_{i_0 j_0} \alpha^{-1} (a_{i_0 j_0} \cdot x_{j_0}) \geqslant x_{j0}$$

Hence $\underset{\sim}{X} \subset \underset{\sim}{K}$.

Inference 2.2. If we have solution in $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B}$, then K^T is the maximum solution of it.

Proof: $\underset{\sim}{X} \circ \underset{\sim}{R} = \underset{\sim}{B} \Longleftrightarrow \underset{\sim}{R}^T \circ \underset{\sim}{X}^T = \underset{\sim}{B}^T \Rightarrow \underset{\sim}{X}^T \subset [(R^T)^T \alpha^{-1} \underset{\sim}{B}^T] = \underset{\sim}{K}$, and $\underset{\sim}{R}^T \circ \underset{\sim}{K} = B^T \Longleftrightarrow$ $\underset{\sim}{X} \subset \underset{\sim}{K}^T$ and $\underset{\sim}{K}^T \circ \underset{\sim}{R} = \underset{\sim}{B}$.

where α^{-1} reprents the compround operation of α^{-1}.

So, the solution introduced in this paper is suitable for the inverse problem of multifaction evaluation.

Ⅲ. The Research for Minimum Solution and Decision for Solvability

Definition 3.1. Stipulate

$$a_{ij}^* = a_{ij} \beta^{-1} b_i \triangleq \begin{cases} k_j, & a_{ij} k_j = b_i \\ 0, & \text{others} \end{cases} \tag{3.1}$$

If k_j is defined by Definition 2.1, it's impossible to make $a_{ij} k_j > b_i$. Then we can get a definition equal to Definition 3.1.

Definition 3.2. Stipulate

$$a_{ij}^* = a_{ij} \beta^{-1} b_i \triangleq \begin{cases} k_j, & a_{ij} k_j = b_i \\ 0, & a_{ij} k_j < b_i \end{cases} \tag{3.2}$$

Definition 3.3. Matrix $\underset{\sim}{A}^* = (a_{ij}^*)$, as nonzero element is the element of solution $\underset{\sim}{K}$, then we call $\underset{\sim}{A}^*$ matrix of candidate solution in (2.1) and the set of each row element in $\underset{\sim}{A}^*$ is called row element set, which writes as follows:

$$\underset{\sim}{A}_i^* = \left(\frac{a_{i1}^*}{x_1} + \frac{a_{i2}^*}{x_2} + \cdots + \frac{a_{im}^*}{x_m} \right) \quad (i \leqslant n)$$

Definition 3.4. Stipulate operator:

$$P \triangleq \left(\Pi \frac{r_i}{x_i}\right)\left(\Sigma \frac{r_j}{x_j}\right) = \begin{cases} \Pi \dfrac{r_i}{x_i} & \text{If } \exists_i = j \\ \text{all multiplication of sum,} & \text{otherwise} \end{cases}$$

Proposition 3.1. $\bigcap_{i \leqslant n} \underset{\sim}{A}_i^* \overset{P}{\Longleftrightarrow} \underset{\sim}{R}^*$

where $\underset{\sim}{R}^* = \sum_{i \leq n} \underset{\sim}{R}_i^* = \sum_i (\prod_i \frac{r_i}{x_i})$. and r_i is one of K_j.

Proof: From Definition 3.4 and by law of set operation, It's easy to obtain: $\bigcap_{i \leq 1} \underset{\sim}{A}_i^* \Longleftrightarrow \sum_i (\prod_i \frac{r_i}{x_i})$.

As $a_{ij}^* = 0$ is omitted in the course of P operation and also nonzero-repeated removable element a_{ij} has to be rejected in the application of absorptive law and so on, hence reserve r_i is one of K_j.

Theorem 3.1 Let $R \neq \phi$. $\underset{\sim}{R}_i^* \in R$ is minimum solution of (2.1) when $a_{ij} K_j = b_i$.

Proof: As $R \neq \phi$, then $\{j;\ a_{ij} K_j = b_i\} \neq \phi$, $(j \in m)$. Hence when $\underset{\sim}{A} \circ \underset{\sim}{R}_i^* = (b_i';\ j \leq m)$, we know the following from Definition 3.4:

$b_i' = \bigvee_{i=1}^{n} (a_{ij} \cdot r_j) \xleftrightarrow{P} \bigvee_{i=1}^{n} (a_{ij} \cdot K_j) = b_i \quad (i \leq n)$.

So $\underset{\sim}{R}_i^*$ is the solution of (2.1). that is a minimum solution. Otherwise if we have another $\underset{\sim}{R}_i^* \subset \underset{\sim}{R}$, there exists (i_0, j_0) such that $r_{j_0}' < r_{j_0}$, then

$\bigvee_{i=1}^{n} (a_{ij} r_j') = a_{i_0 j_0} \cdot r_{j_0}' < a_{i_0 j_0} r_{j_0}$
$= \bigvee_{i=1}^{n} (a_{ij} \cdot r_j) = b_i \quad (i \leq n)$

Contradictory! Hence $\underset{\sim}{R}_i^*$ is the minimum solution for (2.1).

Theorem 3.2. $R \neq \Phi \Longleftrightarrow$. Each row of $\underset{\sim}{A}^*$ has at least a nonzero element.

Proof: "$\Rightarrow$" $R \neq \Phi \xRightarrow{\text{from Theorem 2.1}} a_{ij} x_j \leq b_i$ $(i \leq n, j \leq m)$. And for each i there exists j_i such that $a_{ij_i} \cdot x_{j_i} = b_i$. Then, $K_{j_i} \in R$, so that $a_{ij_i} \cdot K_{j_i} = b_i$ $(i \leq n)$. From Definition 3.2, we know there exists at least a nonzero element in each row of $\underset{\sim}{A}^*$.

"$\Leftarrow$" If there exists at least a nonzero element in each row of $\underset{\sim}{A}^*$, we might as well let $a_{ij_0}^* = K_{j_0} \neq 0$, $(i \leq n)$ while other $a_{ij}^* = 0$. From Definition 3.2, for each i, $a_{ij_0} K_{j_0} = b_i$, $a_{ij} K_j < b_i$ $(j \neq j_0)$. Obviously, it satisfies the sufficient condition in Theorem 2.1. Hence there exists solution in (2.1).

If $\underset{\sim}{R}_i^*$ $(i \leq n)$ is used, $\underset{\sim}{K}$ represents minimum solution and maximum solution, then common solution of (2.1) is:

$R = (\bigcup_i \underset{\sim}{R}_i^*) \cup \underset{\sim}{K} = \underset{\sim}{X} \cup \underset{\sim}{K}$.

Obviously $\underset{\sim}{K}$ is unique, but $\underset{\sim}{K}_i^*$ may not.

IV Application Example

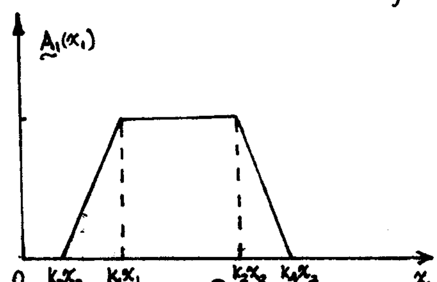
practically, section 2 and 3 have given the solution methord for (2.1), the calculation of which is to be exponded by example.

The next table shows the commodity bought-sold by five stores in the suburb of the city:

	X_1	X_2	X_3	X_4	X_5	$\frac{X_3}{X_2}$	$\frac{X_4}{X_2}$	R_1
y_1	1285	550	25	20	65	0.045	0.036	490
y_2	600	250	14	-0.8	91	0.056	-0.0032	262
y_3	680	408	17	10	82	0.042	0.025	401
y_4	472	438.6	21.5	-3.1	106	0.049	-0.0071	480
y_5	660	367.5	19.8	0.8	72	0.054	0.002	378

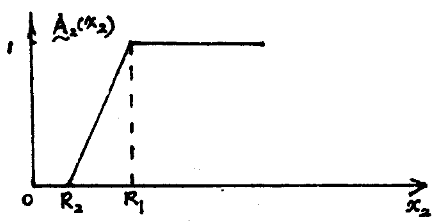
By statistics, the evalution item is x_1 for purchase, x_2 for sale, x_3 for expence, x_4 for benefits (ten thousand a unit), x_5 for fund turnover (one day a unit), and the membership function attained in economical benefits is as follows:

$$(I)\quad \underset{\sim}{A}_1(x_1)=\begin{cases} 0 & 0\le x_1< k_3x_2 \\ \frac{x_1-k_3x_2}{(k_1-k_3)x_2} & k_3x_2\le x_1\le k_1x_2 \\ 1 & k_1x_2<x_1<k_2x_2 \\ \frac{x_1-k_4x_2}{(k_2-k_4)x_2} & k_2x_2\le x_1\le x_4x_2 \\ 0 & k_4x_2<x_1 \end{cases}$$



(I)

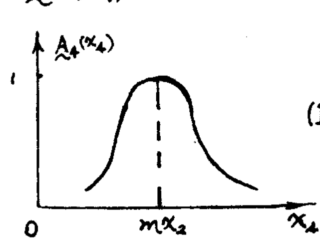
$$(II)\quad \underset{\sim}{A}_2(x_2)=\begin{cases} 1 & R_1<x_2 \\ \frac{x_2-R_1}{R_1-R_2} & R_2\le x_2\le R_1 \\ 0 & x_2<R_2 \end{cases}$$



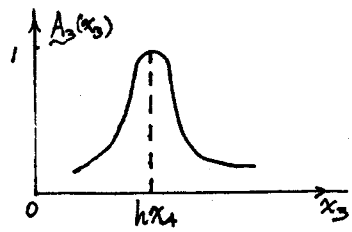
(II)

$$(III)\quad \underset{\sim}{A}_3(x_3)=\frac{1}{1+1000\left(\frac{x_3}{x_2}-\frac{hx_4}{x_2}\right)^2},\quad x_3\in R^+$$

$$(IV)\quad \underset{\sim}{A}_4(x_4)=e^{-\left(\frac{x_4}{x_2}\%-m\right)^2},\quad x_4\in R$$

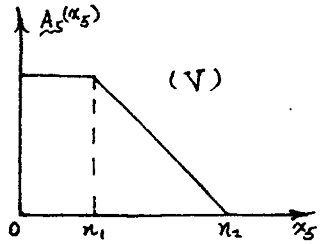


(IV)



(III)

$$(\mathrm{V})\quad \underset{\sim}{A}_5(x_5)=\begin{cases} 1 & 0\leqslant x_5<n_1 \\ \dfrac{x_5-n_1}{n_1-n_2} & n_1\leqslant x_5\leqslant n_2,\ x_5\in[0,365] \\ 0 & n_2<x_5\leqslant 365 \end{cases}$$



According to statistics material, it is proper to select $K_1=1.8$, $K_2=2.2$, $K_3=1.5$, $K_4=2.5$, $h=1$, $m=3.2$, $n_1=62$, $n_2=88$, $R_2=90\%\cdot R_1$ (R_1 represents the former year's sales volumn). Now it is known that the evaluation by experts for five stores in economical benefits is as follows:

$$\underset{\sim}{B}=(\overset{y_1}{0.782},\ \overset{y_2}{0.378},\ \overset{y_3}{0.7},\ \overset{y_4}{0.2},\ \overset{y_5}{0.49})^T$$

Now ~~try~~ to determine influencing degree of each singal target to the whole economical benefits.

Let influencing factor be $\underset{\sim}{X}=(x_1,x_2,x_3,x_4,x_5)^T$, and replace the data in the table and parameter above separately in (I)~(V), then calculate $\underset{\sim}{A}$, Hence Fuzzy relative equations corresponding (1.1) is as follows:

$$\begin{pmatrix} 0.6 & 1 & 0.93 & 0.85 & 0.12 \\ 0.3 & 0.54 & 0.22 & 4\times10^{-6} & 0 \\ 0.56 & 1 & 0.78 & 0.61 & 0.77 \\ 0 & 0.14 & 0.24 & 2.3\times10^{-7} & 0 \\ 0.99 & 0.7 & 0.27 & 1.2\times10^{-4} & 0.38 \end{pmatrix}\circ\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}=\begin{pmatrix} 0.782 \\ 0.378 \\ 0.7 \\ 0.2 \\ 0.49 \end{pmatrix}$$

1° Make augmented matrix $(\underset{\sim}{A}\,|\,\underset{\sim}{B})$, and arrange it in standardization.

$$\left(\begin{array}{ccccc|c} 0.6 & 1 & 0.93 & 0.85 & 0.12 & 0.782 \\ 0.56 & 1 & 0.78 & 0.61 & 0.77 & 0.7 \\ 0.99 & 0.7 & 0.27 & 1.2\times10^{-4} & 0.38 & 0.49 \\ 0.3 & 0.54 & 0.22 & 4\times10^{-6} & 0 & 0.378 \\ 0 & 0.14 & 0.24 & 2.3\times10^{-7} & 0 & 0.2 \end{array}\right)$$

2° From (2.2) and (2.3), we obtain:

$$\left(\begin{array}{ccccc|c} 0.49 & 0.7 & 0.83 & 0.92 & 0.91 & k_j \\ \hline 1 & 0.782 & 0.84 & 0.92 & 1 & 0.782 \\ 1 & 0.7 & 0.9 & 1 & 0.91 & 0.7 \\ 0.49 & 0.7 & 1 & 1 & 1 & 0.49 \\ 1 & 0.7 & 1 & 1 & 1 & 0.378 \\ 1 & 1 & 0.83 & 1 & 1 & 0.2 \end{array}\right)$$

3° From (3.2), $\underset{\sim}{A}^*$ is obtained as follows:

$$\underset{\sim}{A}^*=\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0.92 & 0 \\ 0 & 0.7 & 0 & 0 & 0.91 \\ 0.49 & 0.7 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.83 & 0 & 0 \end{pmatrix} \end{array}$$

From the decision of Theorem 3.4, there exists solution in the equations.

4° Calculate: $\underset{\sim}{A_1}^* \cap \underset{\sim}{A_2}^* \cap \underset{\sim}{A_3}^* \cap \underset{\sim}{A_4}^* \cap \underset{\sim}{A_5}^*$

$$= \frac{0.92}{x_4} \cdot \left(\frac{0.7}{x_2} + \frac{0.91}{x_5}\right) \cdot \left(\frac{0.49}{x_1} + \frac{0.7}{x_2}\right) \cdot \frac{0.7}{x_2} \cdot \frac{0.83}{x_3} \xrightarrow{P} \frac{0.92}{x_4} \cdot \left(\frac{0.7}{x_2} + \frac{0.91}{x_5}\right) \cdot \frac{0.7}{x_2} \cdot \frac{0.83}{x_3}$$

$$= \frac{0.92}{x_4} \cdot \frac{0.7}{x_2} \cdot \frac{0.83}{x_3}$$

So there exists minimum solution in relative equations, that is minimal solution.

$\underset{\sim}{R_1}^* = (0,\ 0.7,\ 0.83,\ 0.92,\ 0)^T$, where its maximum solution is:

$K = (0.49,\ 0.7,\ 0.83,\ 0.92,\ 0.91)^T$, Then

$$R = ([0, 0.49],\ 0.7,\ 0.83,\ 0.92,\ [0, 0.91])^T \tag{4.1}$$

Note: If we come across $r_i = r_j$ when in the application of absorptive law, we have $\frac{r_i}{x_1} + \frac{r_j}{x_2} \cdot \frac{r_i}{x_1} = \frac{r_i}{x_1}$

From <4.1>, the profit influences the economical benefits most largely in the stores and the relative degree is highest, and flexible room is very small. Sale and and expense correlates with economical benefits closely. Though large sale and low expense are needed, yet attention must be paid to the suitable level of purchase and sale and to the appropriate rate of expense and profit. Purchase and fund turnover can be separately changed between [0, 0.49] and [0, 0.91] freely. In fact, more sale, more profit, if enlarge purchase, moving fund will be occupied and fund turnover will slow, which will affect sale. If expense is too low, regular purchase and sale will be affected, then sale decreases and profit will cut down. Besides, it is not noticeable that purchase and fund turnover affect the store in economical benefits, which proves calculation theoretically tallies with practical regularity.

If we take the unique influence factor, a middle point of interval number in (4.1) is easily taken. Then we obtain $R = (0.25, 0.7, 0.83, 0.92, 0.45)^T$.

The M-PFRE is extensively applied in Fuzzy Logic, Operations Research, and also in expert system and economical administration and has deeply practical background and it is worth deeply searching in theory and practice.

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