FUZZY DISTANCE AND LIMIT OF FUZZY COMPLEX NUMBERS

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Abstract

In this paper, we'll introduce three concepts of fuzzy complex number, fuzzy distance and limit of fuzzy complex numbers, and give some elementary properties of the fuzzy complex number and fuzzy distance and limit of the fuzzy complex numbers.

Keywords: Fuzzy number, Fuzzy complex number, Fuzzy distance.

Introduction

J. Buckley, R. Li, S. Yuan, B. Li and W. Fu etc. [2, 3] have done some work about fuzzy complex numbers. They have introduced the concept of fuzzy complex number and given some properties of fuzzy complex number. In this paper, we'll define fuzzy complex number anew and introduce two concepts of fuzzy distance and limit of fuzzy complex numbers. We'll see that the fuzzy distance of two fuzzy complex numbers is a fuzzy number and is also the extension of the fuzzy distance of two fuzzy numbers, this is to say the fuzzy distance of two fuzzy complex numbers is the fuzzy distance of two fuzzy numbers, if two fuzzy complex numbers are two fuzzy numbers. We'll obtain all the properties similar to those of the fuzzy distance and limit of fuzzy numbers[1].

The paper is divided into three sections. In Section 1, after introducing the concept of fuzzy complex number, we discuss some properties of fuzzy complex number, and introduce four fundamental operations of fuzzy complex numbers and the relation " $\leq$ " of fuzzy complex numbers, and define the least upper bound and the greatest lower bound of a set of fuzzy complex numbers, and give an expression to each of them.

In Section 2, we introduce the fuzzy distance of two fuzzy complex numbers, which possesses all the properties of the fuzzy distance of two fuzzy numbers.

In Section 3, we introduce the limit of the sequence of fuzzy complex numbers and obtain all results similar to those of the limit of the sequence of fuzzy numbers.

We'll discuss some important theorems of fuzzy complex numbers in other papers.

1. Basic Definitions and Properties of Fuzzy Complex Numbers

Let C be the set of all complex numbers and R all real numbers and $F^*(R)$ all fuzzy numbers on R[1].

<u>Definition 1.1.</u> Let $\underline{a}$, $\underline{b} \in F^*(R)$, we define

$$(\underline{a}, \underline{b}) : C \longrightarrow [0, 1]$$

$$x+iy \longmapsto \underline{a}(x) \wedge \underline{b}(y),$$

then$(\underline{a}, \underline{b})$ is called a fuzzy complex number on C, $\underline{a}$ is called real part of $(\underline{a}, \underline{b})$ and $\underline{b}$ is called imaginary part of $(\underline{a}, \underline{b})$, we write $c^* = (\underline{a}, \underline{b})$, $\underline{a} = \mathrm{Re}\ c^*$, $\underline{b} = \mathrm{Im}\ c^*$.

Let $F^*(C)$ be the set of all fuzzy complex numbers on C.

If we define $c(z)$ by

$$c(z) = \begin{cases} 1 & \text{iff } x = a,\ y = b; \\ 0 & \text{iff } x \neq a \text{ or } y \neq b, \end{cases}$$

for every $c \in C$, with $c = (a, b)$, $z = (x, y)$, then $c \in F^*(C)$.

Proposition 1.1. For every $c^* \in F^*(C)$, c^* is normal, i.e. there exists $z = (x, y) \in C$ such that

$$c^*(z) = 1.$$

Proof. Since $c^* \in F^*(C)$, then there exist $\underline{a}$, $\underline{b} \in F^*(R)$ such that $c^* = (\underline{a}, \underline{b})$, from $\underline{a}$, $\underline{b}$ are normal, it follows that there exist $x, y \in R$ such that $\underline{a}(x) = 1$, $\underline{b}(y) = 1$. Let $z = (x, y) \in C$, therefore

$$c^*(z) = \underline{a}(x) \wedge \underline{b}(y) = 1.$$

this is to say c^* is normal.

Proposition 1.2. For every $c^* \in F^*(C)$, every $\lambda \in (0, 1]$, λ-cross section[4] of c^*

$$c_\lambda = \{ z;\ c^*(z) \geq \lambda \}$$

is a closed rectangle region

$$\{(x, y);\ (\mathrm{Re}\ c^*)_\lambda^- \leq x \leq (\mathrm{Re}\ c^*)_\lambda^+,\ (\mathrm{Im}\ c^*)_\lambda^- \leq y \leq (\mathrm{Im}\ c^*)_\lambda^+\}$$

with $(\mathrm{Re}\ c^*)_\lambda^{-(+)}$ $((\mathrm{Im}\ c^*)_\lambda^{-(+)})$ denote left(right) endpoint of λ-cross section of $\mathrm{Re}\ c^*(\mathrm{Im}\ c^*)$.

Proof. Since $c^* \in F^*(C)$, then $\mathrm{Re}\ c^*$, $\mathrm{Im}\ c^* \in F^*(R)$, therefore λ-cross sections of them are closed intervals $[(\mathrm{Re}\ c^*)_\lambda^-, (\mathrm{Re}\ c^*)_\lambda^+]$,

$[(\mathrm{Im}\ c^*)_\lambda^-\ ,\ (\mathrm{Im}\ c^*)_\lambda^+]$, it follows that

$$\begin{aligned} c_\lambda &= \{(x\ ,\ y);\ c^*(x\ ,\ y) \geq \lambda\} \\ &= \{(x\ ,\ y);\ (\mathrm{Re}\ c^*)(x) \wedge (\mathrm{Im}\ c^*)(y) \geq \lambda\} \\ &= \{(x\ ,\ y);\ (\mathrm{Re}\ c^*)(x) \geq \lambda \text{ and } (\mathrm{Im}\ c^*)(y) \geq \lambda\} \\ &= \{(x\ ,\ y);\ (\mathrm{Re}\ c^*)_\lambda^- \leq x \leq (\mathrm{Re}\ c^*)_\lambda^+ \text{ and} \\ &\qquad (\mathrm{Im}\ c^*)_\lambda^- \leq y \leq (\mathrm{Im}\ c^*)_\lambda^+\}, \end{aligned}$$

this is to say c_λ is a closed rectangle region.

<u>Proposition 1.3.</u> Every $c^* \in F^*(C)$ is a convex fuzzy set[4].

Proof. For every $\lambda \in [0\ ,\ 1]$, $z_1 = (x_1\ ,\ y_1)$, $z_2 = (x_2\ ,\ y_2) \in C$, since $c^* \in F^*(C)$, then there exist $\underline{a}$, $\underline{b} \in F^*(R)$ such that $c^* = (\underline{a}\ ,\ \underline{b})$, it follows, by using $\underline{a}$, $\underline{b}$ are convex fuzzy sets that

$$\underline{a}(\lambda x_1 + (1-\lambda)x_2) = \underline{a}(x_1) \wedge \underline{a}(x_2);$$

$$\underline{b}(\lambda y_1 + (1-\lambda)y_2) = \underline{b}(y_1) \wedge \underline{b}(y_2),$$

therefore

$$\begin{aligned} &c^*(\lambda z_1 + (1-\lambda)z_2) \\ &= c^*(\lambda(x_1\ ,\ y_1) + (1-\lambda)(x_2\ ,\ y_2)) \\ &= c^*((\lambda x_1 + (1-\lambda)x_2)\ ,\ (\lambda y_1 + (1-\lambda)y_2)) \\ &= \underline{a}(\lambda x_1 + (1-\lambda)x_2) \wedge \underline{b}(\lambda y_1 + (1-\lambda)y_2) \\ &\geq \underline{a}(x_1) \wedge \underline{a}(x_2) \wedge \underline{b}(y_1) \wedge \underline{b}(y_2) \\ &= (\underline{a}(x_1) \wedge \underline{b}(y_1)) \wedge (\underline{a}(x_2) \wedge \underline{b}(y_2)) \\ &= c^*(z_1) \wedge c^*(z_2), \end{aligned}$$

this is to say c^* is a convex fuzzy set.

<u>Proposition 1.4.</u> For every $c^* \in F^*(C)$ and every λ_1, $\lambda_2 \in$

$(0, 1]$, $\lambda_1 \leq \lambda_2$, then

$$c_{\lambda_1} \supset c_{\lambda_2} .$$

Proof. Obvious.

Definition 1.2. For every c_1^*, $c_2^* \in F^*(C)$, we define

1) $c_1^* + c_2^* \triangleq (\text{Re } c_1^* + \text{Re } c_2^* , \text{Im } c_1^* + \text{Im } c_2^*)$;

2) $c_1^* - c_2^* \triangleq (\text{Re } c_1^* - \text{Re } c_2^* , \text{Im } c_1^* - \text{Im } c_2^*)$;

3) $c_1^* \wedge c_2^* \triangleq (\text{Re } c_1^* \wedge \text{Re } c_2^* , \text{Im } c_1^* \wedge \text{Im } c_2^*)$;

4) $c_1^* \vee c_2^* \triangleq (\text{Re } c_1^* \vee \text{Re } c_2^* , \text{Im } c_1^* \vee \text{Im } c_2^*)$;

5) $c_1^* \cdot c_2^* \triangleq (\text{Re } c_1^* \cdot \text{Re } c_2^* , \text{Im } c_1^* \cdot \text{Im } c_2^*)$;

6) $c_1^* \div c_2^* \triangleq (\text{Re } c_1^* \div \text{Re } c_2^* , \text{Im } c_1^* \div \text{Im } c_2^*)$, with Re c_2^* (Im c_2^*) is positive fuzzy number or negative fuzzy number[4];

7) $c \cdot c^* \triangleq (a \cdot \text{Re } c^* , b \cdot \text{Im } c^*)$, $c = (a , b)$.

It is easy to see that if c_1^*, $c_2^* \in F^*(C)$, then $c_1^* + c_2^*$, $c_1^* - c_2^*$, $c_1^* \wedge c_2^*$, $c_1^* \vee c_2^*$, $c_1^* \cdot c_2^*$, $c_1^* \div c_2^* \in F^*(C)$.

Definition 1.3. For c_1^*, $c_2^* \in F^*(C)$, we say that $c_1^* \leq c_2^*$, if

$$\text{Re } c_1^* \leq \text{Re } c_2^* \quad \text{and} \quad \text{Im } c_1^* \leq \text{Im } c_2^* .$$

We say that $c_1^* < c_2^*$, if $c_1^* \leq c_2^*$ and Re $c_1^* <$ Re c_2^* or Im $c_1^* <$ Im c_2^*. We say that $c_1^* = c_2^*$, if $c_1^* \leq c_2^*$ and $c_2^* \leq c_1^*$.

Proposition 1.5. $F^*(C)$ is dense[1].

Definition 1.4. For every $c^* \in F^*(C)$, if Re $c^* = \underline{\infty}$ or Im $c^* = \underline{\infty}$, then c^* is called fuzzy complex infinity, write ∞^*.

Definition 1.5. Let $A \subset F^*(C)$, if there exists $M^* \in F^*(C)$, $M^* \neq \infty^*$, such that

$$c^* \leq M^* \quad \text{for every } c^* \in A,$$

then A is said to have an upper bound M^*. Similarly, if there exists $m^* \in F^*(C)$, $m^* \neq \infty^*$, such that

$$m^* \leq c^* \quad \text{for every } c^* \in A,$$

then A is said to have a lower bound m^*.

A set with both upper and lower bounds is said to be bounded.

Definition 1.6. Let $A \subset F^*(C)$, $M^* \in F^*(C)$ is called the least upper bound of A, if M^* has the properties:

1) Whenever $c^* \in A$, we have $c^* \leq M^*$;

2) For any $\varepsilon > 0$, there exists at least one $c^* \in A$ such that

$$M^* < (\mathrm{Re}\, c^* + \varepsilon, \mathrm{Im}\, c^* + \varepsilon),$$

we write $M^* = \sup A$.

Similarly, we introduce

Definition 1.7. Let $A \subset F^*(C)$, $m^* \in F^*(C)$ is called the greatest lower bound of A, if m^* has the properties:

1) Whenever $c^* \in A$, we have $m^* \leq c^*$;

2) For any $\varepsilon > 0$, there exists at least one $c^* \in A$ such that

$$(\mathrm{Re}\, c^* - \varepsilon, \mathrm{Im}\, c^* - \varepsilon) < m^*,$$

we write $m^* = \inf A$.

Theorem 1.1. Let $A \subset F^*(C)$, if A has the least upper bound and the greatest lower bound, then

$$\sup A = (\sup_{c^* \in A} \mathrm{Re}\, c^*, \sup_{c^* \in A} \mathrm{Im}\, c^*)$$

and

$$\inf A = (\inf_{c^* \in A} \mathrm{Re}\, c^*, \inf_{c^* \in A} \mathrm{Im}\, c^*).$$

Proof. Obvious.

2. Fuzzy Distance of Fuzzy Complex Numbers and its Properties

Definition 2.1. Afuzzy distance of fuzzy complex numbers is a function $\rho^*: (F^*(C)\ ,\ F^*(C)) \longrightarrow F^*(R)$ with the properties:

1) $\rho^*(c_1^*\ ,\ c_2^*) \geq 0$, $c_1^* = c_2^*$ iff $\rho^*(c_1^*\ ,\ c_2^*) = 0$;

2) $\rho^*(c_1^*\ ,\ c_2^*) = \rho^*(c_2^*\ ,\ c_1^*)$;

3) Whenever $c_3^* \in F^*(C)$, we have

$$\rho^*(c_1^*\ ,\ c_2^*) \leq \rho^*(c_1^*\ ,\ c_3^*) + \rho^*(c_3^*\ ,\ c_2^*).$$

If ρ^* is the fuzzy distance of fuzzy complex numbers, we call $(C\ ,\ F^*(C)\ ,\ \rho^*)$ a fuzzy distance space.

In the following, we introduce a function ρ^*, which plays a key role in the theory of fuzzy complex numbers.

We define

$$(*) \qquad \rho^*(c_1^*\ ,\ c_2^*) = \underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_2^*) \vee \underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_2^*),$$

for every $c_1^*, c_2^* \in F^*(C)$.

Theorem 2.1. $\rho^*(c_1^*\ ,\ c_2^*)$ defined by the equality $(*)$ is the fuzzy distance of fuzzy complex numbers.

Proof. 1) For every $c_1^*\ ,\ c_2^* \in F^*(C)$, then $\underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_2^*) \geq 0$, $\underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_2^*) \geq 0$, therefore

$$\rho^*(c_1^*\ ,\ c_2^*) \geq 0.$$

Suppose that $\rho^*(c_1^*\ ,\ c_2^*) = 0$, then

$$\underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_2^*) \vee \underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_2^*) = 0,$$

therefore

$$\underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_2^*) = 0 \text{ and } \underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_2^*) = 0,$$

it yields that

$$\mathrm{Re}\ c_1^* = \mathrm{Re}\ c_2^* \text{ and } \mathrm{Im}\ c_1^* = \mathrm{Im}\ c_2^*,$$

this is to say $c_1^* = c_2^*$.

Otherwise, suppose that $c_1^* = c_2^*$, then $\mathrm{Re}\ c_1^* = \mathrm{Re}\ c_2^*$ and $\mathrm{Im}\ c_1^* = \mathrm{Im}\ c_2^*$, therefore

$$\underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_2^*) = 0 \text{ and } \underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_2^*) = 0,$$

thus

$$\rho^*(c_1^*\ ,\ c_2^*) = 0.$$

2) Obvious.

3) For every $c_3^* \in F^*(C)$, we have

$$\begin{aligned}
\underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_2^*) &\le \underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_3^*) + \underline{\rho}(\mathrm{Re}\ c_3^*\ ,\ \mathrm{Re}\ c_2^*) \\
&\le \underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_3^*) \vee \underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_3^*) \\
&\quad + \underline{\rho}(\mathrm{Re}\ c_3^*\ ,\ \mathrm{Re}\ c_2^*) \vee \underline{\rho}(\mathrm{Im}\ c_3^*\ ,\ \mathrm{Im}\ c_2^*) \\
&= \rho^*(c_1^*\ ,\ c_3^*) + \rho^*(c_3^*\ ,\ c_2^*);
\end{aligned}$$

$$\begin{aligned}
\underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_2^*) &\le \underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_3^*) + \underline{\rho}(\mathrm{Im}\ c_3^*\ ,\ \mathrm{Im}\ c_2^*) \\
&\le \underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_3^*) \vee \underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_3^*) \\
&\quad + \underline{\rho}(\mathrm{Re}\ c_3^*\ ,\ \mathrm{Re}\ c_2^*) \vee \underline{\rho}(\mathrm{Im}\ c_3^*\ ,\ \mathrm{Im}\ c_2^*) \\
&= \rho^*(c_1^*\ ,\ c_3^*) + \rho^*(c_3^*\ ,\ c_2^*),
\end{aligned}$$

therefore

$$\begin{aligned}
\rho^*(c_1^*\ ,\ c_2^*) &= \underline{\rho}(\mathrm{Re}\ c_1^*\ ,\ \mathrm{Re}\ c_2^*) \vee \underline{\rho}(\mathrm{Im}\ c_1^*\ ,\ \mathrm{Im}\ c_2^*) \\
&\le \rho^*(c_1^*\ ,\ c_3^*) + \rho^*(c_3^*\ ,\ c_2^*).
\end{aligned}$$

<u>Theorem 2.2.</u> Whenever $c_1^*, c_2^*, c_3^*, c_4^* \in F^*(C)$, $c \in C$, we have

1) $\rho^*(c_1^* + c_3^*, c_2^* + c_3^*) = \rho^*(c_1^*, c_2^*)$;

2) $\rho^*(c_1^* - c_3^*, c_2^* - c_3^*) = \rho^*(c_1^*, c_2^*)$;

3) $\rho^*(c_3^* - c_1^*, c_3^* - c_2^*) = \rho^*(c_1^*, c_2^*)$;

4) $(|\mathrm{Re}\ c| \wedge |\mathrm{Im}\ c|)\rho^*(c_1^*, c_2^*)$

$\leq |\mathrm{Re}\ c|\underline{\rho}(\mathrm{Re}\ c_1^*, \mathrm{Re}\ c_2^*) \vee |\mathrm{Im}\ c|\underline{\rho}(\mathrm{Im}\ c_1^*, \mathrm{Im}\ c_2^*)$

$= \rho^*(c \cdot c_1^*, c \cdot c_2^*) \leq (|\mathrm{Re}\ c| \vee |\mathrm{Im}\ c|)\rho^*(c_1^*, c_2^*)$;

5) If $c_1^* \leq c_2^* \leq c_3^*$, then

$\rho^*(c_1^*, c_2^*) \leq \rho^*(c_1^*, c_3^*)$ and $\rho^*(c_2^*, c_3^*) \leq \rho^*(c_1^*, c_3^*)$;

6) If $c_1^* \leq c_3^* \leq c_2^*$ and $c_1^* \leq c_4^* \leq c_2^*$, then

$$\rho^*(c_3^*, c_4^*) = 2 \cdot \rho^*(c_1^*, c_2^*).$$

Proof. The proof is obvious.

3. Limit of the Sequence of Fuzzy Complex Numbers

Definition 3.1. Let $\{c_n^*\} \subset F^*(C)$, $c^* \in F^*(C)$, $\{c_n^*\}$ is said to converge to c^* in fuzzy distance ρ^*, denoted by

$$\lim_{n\to\infty} c_n^* = c^* \text{ or } c_n^* \longrightarrow c^* (n \longrightarrow \infty),$$

if for arbitrary $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$\rho^*(c_n^*, c^*) < \varepsilon \quad \text{as } n \geq N.$$

Theorem 3.1.

$$\lim_{n\to\infty} c_n^* = c^* \text{ iff } \lim_{n\to\infty}(\mathrm{Re}\ c_n^*) = \mathrm{Re}\ c^* \text{ and } \lim_{n\to\infty}(\mathrm{Im}\ c_n^*) = \mathrm{Im}\ c^*.$$

Proof. Obvious.

Theorem 3.2. Let $\{c_n^*\}$, $\{d_n^*\} \subset F^*(C)$, $c^*, d^* \in F^*(C)$, $c \in C$, if $\lim_{n\to\infty} c_n^* = c^*$ and $\lim_{n\to\infty} d_n^* = d^*$, then

1) $\lim_{n\to\infty} (c_n^* + d_n^*) = c^* + d^*$;

2) $\lim_{n\to\infty} (c_n^* - d_n^*) = c^* - d^*$;

3) $\lim_{n\to\infty} (c \cdot c_n^*) = c \cdot c^*$.

Proof. 1), 2) Obvious.

3) Since $\lim_{n\to\infty} c_n^* = c^*$, then by using theorem 3.1 that

$$\lim_{n\to\infty}(\mathrm{Re}\ c_n^*) = \mathrm{Re}c^* \quad \text{and} \quad \lim_{n\to\infty}(\mathrm{Im}\ c_n^*) = \mathrm{Im}\ c^*,$$

therefore, for any $\varepsilon > 0$, there exist N_1, $N_2 \geq 0$ such that

$$\underline{\rho}(\mathrm{Re}\ c_n^*\ ,\ \mathrm{Re}\ c^*) < \varepsilon/(|\mathrm{Re}\ c^*| + 1),$$

as $n \geq N_1$, and

$$\underline{\rho}(\mathrm{Im}\ c_n^*\ ,\ \mathrm{Im}\ c^*) < \varepsilon/(|\mathrm{Im}\ c^*| + 1),$$

as $n \geq N_2$. Let $N = \max\{N_1\ ,\ N_2\}$, then we have

$$\begin{aligned} \rho^*(c \cdot c_n^*\ ,\ c \cdot c^*) &= |\mathrm{Re}\ c|\,\underline{\rho}(\mathrm{Re}\ c_n^*\ ,\ \mathrm{Re}\ c^*) \\ &\quad \vee |\mathrm{Im}\ c|\,\underline{\rho}(\mathrm{Im}\ c_n^*\ ,\ \mathrm{Im}\ c^*) \\ &< |\mathrm{Re}\ c|\,\varepsilon/(|\mathrm{Re}\ c| + 1) \vee |\mathrm{Im}\ c|\,\varepsilon/(|\mathrm{Im}\ c| + 1) \\ &\leq \varepsilon\ , \end{aligned}$$

as $n \geq N$. This means that

$$\lim_{n\to\infty} (c \cdot c_n^*) = c \cdot c^*.$$

<u>Theorem 3.3.</u> (Limit uniqueness theorem) If $\lim_{n\to\infty} c_n^* = c^*$, and $\lim_{n\to\infty} c_n^* = d^*$, then

$$c^* = d^*.$$

Proof. Obvious.

Theorem 3.4. Let $\{c_n^*\}$, $\{d_n^*\}$, $\{e_n^*\} \subset F^*(C)$, $c^* \in F^*(C)$, if for every n, $c_n^* \leq d_n^* \leq e_n^*$, and $\lim_{n\to\infty} c_n^* = \lim_{n\to\infty} e_n^* = c^*$, then

$$\lim_{n\to\infty} d_n^* = c^*.$$

Proof. Obvious.

Theorem 3.5. (Boundedness theorem) Let $\{c_n^*\} \subset F^*(C)$, $c^* \neq \infty^*$ $c_n^* \neq \infty^*$, $n = 1, 2, \cdots$, if $\lim_{n\to\infty} c_n^* = c^*$, then there exist M^*, m^* $(\neq \infty^*) \in F^*(C)$ such that

$$m^* \leq c_n^* \leq M^*,$$

for every n.

Proof. Obvious.

Theorem 3.6. (Keeping sign property theorem) Let $\{c_n^*\} \subset F^*(C)$, $c^*, d^* \in F^*(C)$, $\lim_{n\to\infty} c_n^* = c^*$, if for every n,

$$c_n^* \leq d^* (d^* \leq c_n^*),$$

then

$$c^* \leq d^* (d^* \leq c^*).$$

Proof. Obvious.

Theorem 3.7. Let $\lim_{n\to\infty} c_n^* = c^*$ and $\lim_{n\to\infty} d_n^* = d^*$, then

$$\lim_{n\to\infty} \rho^*(c_n^*, d_n^*) = \rho^*(c^*, d^*).$$

Proof. Since $\lim_{n\to\infty} c_n^* = c^*$ and $\lim_{n\to\infty} d_n^* = d^*$, then from theorem 3.1 that

$$\lim_{n\to\infty} (\mathrm{Re}\, c_n^*) = \mathrm{Re}\, c^*, \quad \lim_{n\to\infty} (\mathrm{Im}\, c_n^*) = \mathrm{Im}\, c^*,$$

and

$$\lim_{n\to\infty} (\mathrm{Re}\ d_n^*) = \mathrm{Re}\ d^*, \quad \lim_{n\to\infty} (\mathrm{Im}\ d_n^*) = \mathrm{Im}\ d^*.$$

By using theorem 3.7 of [1] that

$$\lim_{n\to\infty} \underline{\rho}(\mathrm{Re}\ c_n^*, \mathrm{Re}\ d_n^*) = \underline{\rho}(\mathrm{Re}\ c^*, \mathrm{Re}\ d^*)$$

and

$$\lim_{n\to\infty} \underline{\rho}(\mathrm{Im}\ c_n^*, \mathrm{Im}\ d_n^*) = \underline{\rho}(\mathrm{Im}\ c^*, \mathrm{Im}\ d^*),$$

therefore, for any $\varepsilon > 0$, there exist $N_1, N_2 > 0$ such that

$$\underline{\rho}(\mathrm{Re}\ c^*, \mathrm{Re}\ d^*) - \varepsilon \leq \underline{\rho}(\mathrm{Re}\ c_n^*, \mathrm{Re}\ d_n^*) \leq \underline{\rho}(\mathrm{Re}\ c^*, \mathrm{Re}\ d^*) + \varepsilon$$

and

$$\underline{\rho}(\mathrm{Im}\ c^*, \mathrm{Im}\ d^*) - \varepsilon \leq \underline{\rho}(\mathrm{Im}\ c_n^*, \mathrm{Im}\ d_n^*) \leq \underline{\rho}(\mathrm{Im}\ c^*, \mathrm{Im}\ d^*) + \varepsilon,$$

thus, when $n \geq \max\{N_1, N_2\}$, we have

$$(\underline{\rho}(\mathrm{Re}\ c^*, \mathrm{Re}\ d^*) - \varepsilon) \vee (\underline{\rho}(\mathrm{Im}\ c^*, \mathrm{Im}\ d^*) - \varepsilon)$$
$$\leq (\underline{\rho}(\mathrm{Re}\ c_n^*, \mathrm{Re}\ d_n^*)) \vee (\underline{\rho}(\mathrm{Im}\ c_n^*, \mathrm{Im}\ d_n^*))$$
$$= (\underline{\rho}(\mathrm{Re}\ c^*, \mathrm{Re}\ d^*) + \varepsilon) \vee (\underline{\rho}(\mathrm{Im}\ c^*, \mathrm{Im}\ d^*) + \varepsilon),$$

it yields that

$$\rho^*(c^*, d^*) - \varepsilon = \rho^*(c_n^*, d_n^*) = \rho^*(c^*, d^*) + \varepsilon,$$

as $n \geq N$, this means that

$$\lim_{n\to\infty} \rho^*(c_n^*, d_n^*) = \rho^*(c^*, d^*).$$

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