ON THE REDUCTION OF FUZZY NUMBERS TO REAL

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In reducing fuzzy numbers to real numbers (for example, in decision-making) most of the information is of necessity lost. Fortunately, as will be shown below, in going to the Steiner centroids of fuzzy numbers the "structural" information, the one contained in the totality of fuzzy numbers, yet keeps. In particular, the crisp numbers are transformed into their supports' mid-points, and the fuzzy set "small" borrowed from Zadeh yields 3.

INTRODUCTION. Let $0 < r \leq 1$ and $-1 \leq x^* \leq 1$.

A <u>fuzzy number</u> is a fuzzy set such that each its r-level set is a closed interval of the real line and the support is bounded. For details, see /1/. Let M (and N, too) be a fuzzy number, M_r its r-level set and M_0 the support. Note that the (end-point) functions $\inf M_r$ and $\sup M_r$ of the variable r are monotone and bounded.

The <u>support function</u> of M is the function

$$x^* \to M^*(x^*) = \sup\{xx^* : x \in M_{|x^*|}\}.$$

For details, see /2/; in particular, the support functions are upper semicontinuous and functionally bounded.

The Steiner centroid of M is the real number

$$k(M) = \frac{3}{2}\int_0^1 r^2(\inf M_r + \sup M_r)dr;$$

in other words, in terms of support functions,

$$k(M) = \frac{3}{2}\int_{-1}^1 x^*M^*(x^*)dx^*.$$

For the non-fuzzy case in multidimensional Euclidean space, see /3/. See also Supplement. Clearly, $k(M) \in M_0$ and, in the crisp case,

$$k(M) = \frac{1}{2}(\inf M_0 + \sup M_0).$$

As a result, the concentrated numbers, the fuzzy numbers whose supports are singletons, are "fixed points" of $k(.)$.

1. MONOTONICITY. As usual, we write $M \leq N$ iff

$$\inf M_r \leq \inf N_r, \ \sup M_r \leq \sup N_r \ \forall\ r;$$

in other words,

$$M \leq N \iff x^*M^*(x^*) \leq x^*N^*(x^*) \ \forall\ x^*.$$

Hence

$$M \leq N \implies k(M) \leq k(N), \quad \text{i.e. } k(.) \text{ is increasing.}$$

2. ADDITIVITY AND HOMOGENEITY. The **Minkowski sum** of M and N is a fuzzy number M + N such that

$$(M + N)_r = [\inf M_r + \inf N_r,\ \sup M_r + \sup N_r] \ \forall\ r;$$

in other words,

$$(M + N)^*(x^*) = M^*(x^*) + N^*(x^*) \ \forall\ x^*.$$

Hence

$$k(M + N) = k(M) + k(N),$$

i.e. k(.) is additive.

The **product** of M **by a real number**, by a, is a fuzzy number aM such that

$$(aM)_r = \begin{cases} [a\inf M_r,\ a\sup M_r], & \text{if } a \geq 0, \\ [+a\sup M_r,\ +a\inf M_r], & \text{if } a \leq 0, \end{cases} \ \forall\ r;$$

in other words,

$$(aM)^*(x^*) = |a|M^*(\mathrm{sign}(a)x^*) \ \forall\ x^*.$$

Hence

$$k(aM) = ak(M), \quad \text{i.e. } k(.) \text{ is homogeneous.}$$

3. CONTINUITY. The L_p distance, $1 \leq p < \infty$, and the Hausdorff distance between M and N are the non-negative numbers

$$h_p(M, N) = \left(\frac{1}{2}\int_0^1 (|\inf M_r - \inf N_r|^p + |\sup M_r - \sup N_r|^p)\,dr\right)^{1/p}$$

and

$$h_\infty(M, N) = \sup_r \max\left\{|\inf M_r - \inf N_r|, |\sup M_r - \sup N_r|\right\};$$

in other words,

$$h_p(M, N) = \left(\frac{1}{2}\int_{-1}^{1} \frac{1}{|x^*|^p}|M^*(x^*) - N^*(x^*)|^p dx^*\right)^{1/p}$$

and

$$h_\infty(M, N) = \sup_{x^* \neq 0} \frac{1}{|x^*|}|M^*(x^*) - N^*(x^*)|.$$

Hence, via Hölder's inequality,

$$|k(M) - k(N)| \leq C_p h_p(M, N),\ 3 \geq C_p \geq 1,\ 1 \leq p \leq \infty,$$

i.e. k(.) is Lipschitzean.

CONCLUSION. Besides the Steiner centroid, in the non-fuzzy case there are also other centroids worthy of note; see /4/. Unfortunately, as regards the fuzzy case, I have nothing to add here, but I thank the reader in advance for any information and cooperation.

REFERENCES

/1/ O. Kaleva and S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems 12 (1984) 215-229.

/2/ V. N. Bobylev, Support function of a fuzzy set and its characteristic properties, Math. Notes 37 (1985) 281-285.

/3/ R. Schneider, Steiner points of convex bodies, Israel J. Math. 9 (1971) 241-249.

/4/ B. Grünbaum, Measures of symmetry for convex sets, in: V. L. Klee, Ed., Proceedings of Symposia in Pure Mathematics 8 (1963) 233-270.

SUPPLEMENT. For the Steiner centroid to exist it is not necessary that the fuzzy number be support bounded. Consider, for example, the fuzzy set "small" defined in a Zadeh book by the membership function

$$M(x) = \left(1 + \left(\frac{x}{10}\right)^2\right)^{-1}, \quad x = 0, 1, 2, \dots .$$

Redefine it for the other real numbers to the effect that

$$M_r = \left[0,\ 10\left(\frac{1-r}{r}\right)^{1/2}\right] \ \forall\ r.$$

Preserving the definition of Steiner centroid, we obtain

$$k(M) = \frac{3}{2}\int_0^1 r^2 10\left(\frac{1-r}{r}\right)^{1/2} dr = 15B\left(\frac{5}{2}, \frac{3}{2}\right) = 15\frac{\pi}{16} = 2.945\dots .$$

In this sense, a "small integer" means 3.