SOLUTION OF LR-TYPE FUZZY SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

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ABSTRACT

Systems of linear algebraic equations with LR-type fuzzy coefficients are considered in this article. The notion of solution of LR-type fuzzy sistem is discussed. It is shown that in general case the exact solution may not exist, so it is offered to find an approximate one (quasi-solution). It appears that such problem may be reduced to an ordinary (non-fuzzy) non-linear optimization problem. The numerical example of application of this method is provided.

A system of linear algebraic equations is the simplest and the most useful mathematical model for a lot of problems consideredby applied mathematics. In practice, unfortunately, the exact values of coefficients of these systems are not as a rule known. This **uncertainty may** have either probabilistic or non-probabilistic nature. Accordingly, different approaches to the problem and different mathematical tools are needed.

In this article systems of linear algebraic equations whose coefficients and right-hand sides are fuzzy numbers of certain type are considered.

First of all it is necessary to determine arithmetic operations on fuzzy numbers. For this purpose **the** extension principal introduced by Zadeh [6] is used. According to Zade's principal :

$$\mu_{\tilde{x}+\tilde{y}}(z) = \sup_{z=x+y} \min\{\mu_{\tilde{x}}(x), \mu_{\tilde{y}}(y)\} \tag{1a}$$

$$\mu_{\tilde{x}-\tilde{y}}(z) = \sup_{z=x-y} \min\{\mu_{\tilde{x}}(x), \mu_{\tilde{y}}(y)\} \tag{1b}$$

$$\mu_{\tilde{x}\cdot\tilde{y}}(z) = \sup_{z=xy} \min\{\mu_{\tilde{x}}(x), \mu_{\tilde{y}}(y)\} \tag{1c}$$

$$\mu_{\tilde{x}/\tilde{y}}(z) = \sup_{z=x/y} \min\{\mu_{\tilde{x}}(x), \mu_{\tilde{y}}(y)\} \tag{1d}$$

(for more details see [1], [4]).

Evidently it is inconvenient to use formulae (1a)-(1d) in general case. We will use the conception of LR-type fuzzy numbers introduced by Dubois and Prade ([1] -[4]):

Let us consider two continious functions $L : R^1 \to [0;1]$ and $R : R^1 \to [0;1]$ with the following properties :

1) $L(-x) = L(x)$; $R(-x) = R(x)$
2) $L(0) = 1$; $R(0) = 1$
3) L and R are non-icreasing on $[0; +\infty[$
4) $\lim_{x\to\infty} L(x) = 0$; $\lim_{x\to\infty} R(x) = 0$

Definition. A fuzzy number $\tilde{m}$ is said to be an LR-type fuzzy number iff

$$\mu_{\tilde{m}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & x \leq m, \alpha > 0 \\ R\left(\frac{x-m}{\beta}\right), & x > m, \beta > 0 \end{cases}$$

(if $\alpha = \beta = 0$, $\tilde{m}$ is an ordinary (non-fuzzy) number m)

m characterizes the mean value of $\tilde{m}$, while α and β are the left and the right coefficients of "fuzziness" respectively.

Symbollicaly we will write : $\tilde{m} = (m, \alpha, \beta)_{LR}$

Thus an LR-type fuzzy number is determined by the types of functions L, R and by 3 parametrs.

It is clear that in many cases the membership function of an arbitrary fuzzy number may be approximated by membership function of LR-type so in the ugnel we will deal only with LR-type fuzzy numbers.

Arithmetic operations for LR-type fuzzy numbers are much simpler than (1a) - (1d). Indeed, it may be shown ([4]), that

$$(m, \alpha, \beta)_{LR} + (n, \gamma, \delta)_{LR} = (m+n, \alpha+\gamma, \beta+\delta)_{LR} \qquad (2)$$

$$-(m, \alpha, \beta)_{LR} = (-m, \beta, \alpha)_{RL} \qquad (3)$$

$$(m, \alpha, \beta)_{LR} - (n, \gamma, \delta)_{RL} = (m-n, \alpha+\delta, \beta+\gamma)_{LR} \qquad (4)$$

Knowing types of L,R fuctions and calculating respective parametres by (2)-(4) one can simply obtain membership functions for addition and substraction of LR-type fuzzy numbers.

The product $\tilde{m}\cdot\tilde{n}$ will not be in general case an LR-type fuzzy number as well, but the following approximate formulae may be used :

$$(m,\alpha,\beta)_{LR} \cdot (n,\gamma,\delta)_{LR} \approx (mn, \alpha n+\gamma m, \beta n+\delta m)_{LR}, \tilde{m}>0, \tilde{n}>0 \qquad (5a)$$

$$(m,\alpha,\beta)_{LR} \cdot (n,\gamma,\delta)_{RL} \approx (mn, \gamma m-\beta n, \delta m-\alpha n)_{RL}, \tilde{m}>0, \tilde{n}\ \ 0 \qquad (5b)$$

$$(m,\alpha,\beta)_{RL} \cdot (n,\gamma,\delta)_{LR} \approx (mn, \alpha n-\delta m, \beta n-\gamma m)_{RL}, \tilde{m}<0, \tilde{n}>0 \qquad (5c)$$

$$(m,\alpha,\beta)_{LR} \cdot (n,\gamma,\delta)_{LR} \approx (mn, -\beta n-\delta m, -\alpha n-\gamma m)_{RL}, \tilde{m}<0, \tilde{n}<0 \qquad (5d)$$

$$(m,\alpha,\beta)_{LR}^{-1} \approx \left(\frac{1}{m}, \frac{\beta}{m^2}, \frac{\alpha}{m^2}\right)_{RL}, \tilde{m}>0 \qquad (6a)$$

$$(m,\alpha,\beta)_{LR} : (n,\gamma,\delta)_{RL} \approx \left(\frac{m}{n}, \frac{\delta m+\alpha n}{n^2}, \frac{\gamma m+\beta n}{n^2}\right)_{LR}, \tilde{m}>0, \tilde{n}>0 \qquad (6b)$$

Remarks :

1) formulae for multiplication and division depend on the choise of signs of factors $\tilde{m}$ and $\tilde{n}$

2) 0 can not be included in supports of factors, that is
$0 \notin \text{supp}\, \tilde{m};\ 0 \notin \text{supp}\, \tilde{n}$

Let us discuss some aspects of applying these approximate formulae. Let, for example, L = R, $\alpha = \beta$, $\gamma = \delta$, $\tilde{m} = (m, \alpha, \beta)_{LL} > 0$, $\tilde{n} = (n, \gamma, \gamma)_{LL} > 0$. Let us calculate the error made by using (5a), We have (by [5]) :

$$\mu_{\tilde{m}\tilde{n}}(z) = \begin{cases} L(f(z)), & z \leq mn \\ L(-f(z)), & z > mn \end{cases}$$

where $f(z) = \dfrac{\alpha n + \gamma m}{2\alpha\gamma} - \dfrac{1}{2\alpha\gamma}\sqrt{(\alpha n - \gamma m)^2 + 4\alpha\gamma z} = \dfrac{mn - z}{\alpha n + \gamma m} \cdot \dfrac{1}{k(z)}$,

$$k(z) = 0{,}5\left(1 + \left[1 + \frac{4\alpha\gamma(z - mn)}{(n + m)^2}\right]^{1/2}\right) \geq 0{,}5$$

Let $f(z) \approx f^*(z) = \dfrac{mn - z}{\alpha n + \gamma m}$

The relative error of the last approximation is $\varepsilon(z) = 1 - k(z) \leq 0{,}5$. Let $L_+^{-1}(0)$ denotes the minimal positive root of equation $L(x)=0$, while $L_-^{-1}(0)$ denotes the maximal negative one (evidently the set of solutions of the equation $L(x)=0$ is $]-\infty; L_-^{-1}(0)] \cup [L_+^{-1}(0); +\infty[$). It is clear, in view of the properties of L there exist finite $L_+^{-1}(0)$ and $L_-^{-1}(0)$ (since $0 \notin \text{supp}\, \tilde{m}, 0 \notin \text{supp}\, \tilde{n}$) and $L_+^{-1}(0) = -L_-^{-1}(0)$.

Then $\text{supp}\, \tilde{m}\tilde{n} = [mn - L_+^{-1}(0)(\alpha n + \gamma m);\ mn + L_+^{-1}(0)(\alpha n + \gamma m)]$

It is easy to show that $\varepsilon_{max} = \max_z \varepsilon(z) \leq 0,5(1 - \left[1 - \frac{4\alpha\gamma\Delta}{(\alpha|n| + \gamma|m|)^2}\right]^{1/2})$, where $\Delta = L_+^{-1}(0)(\alpha n + \gamma m)$, or in general case $\Delta = L_+^{-1}(0)(\alpha|n| + \gamma|m|)$.

Let us suppose that $\max_x |L'(x)| = M < \infty$. Then

$$\delta(z) = \left|L(f(z)) - L(f^*(z))\right| \leq M \cdot \max_z |\varepsilon(z)| \cdot \max_z |f(z)| \leq M \frac{mn}{\alpha|n| + \gamma|m|} \cdot \frac{\varepsilon_{max}}{1 - \varepsilon_{max}} \quad (7)$$

This estimate is too rough and therefore may be used only for moderate values of M. If M is large or infinite one has to estimate δ_{max} by solving the equation :

$$\delta'(z) = L'(f(z)) f'(z) + L'\left(\frac{mn - z}{\alpha|n| + \gamma|m|}\right) \cdot \frac{1}{\alpha|n| + \gamma|m|} = 0 \quad (8)$$

Now, knowing how to deal with fuzzy numbers, let us discuss the problem of solution of systems of linear algebraic equations with fuzzy coefficients and fuzzy right-hand sides :

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \dots + \tilde{a}_{1n}\tilde{x}_n = \tilde{b}_1 \\ \dots\dots\dots\dots\dots\dots\dots \\ \tilde{a}_{n1}\tilde{x}_1 + \dots + \tilde{a}_{nn}\tilde{x}_n = \tilde{b}_n \end{cases} \quad (9)$$

At the first sight the usual methods such as Kramer's or Gauss can be used. But this is a wrong way. Let us illustrate this for the case $n=1$:

$$\tilde{a}\tilde{x} = \tilde{b} \quad (10)$$

It would be wrong to write the solution of (10) in the "usual" form : $\tilde{x} = \tilde{b}/\tilde{a}$. The problem is that the set of fuzzy numbers is no longer a group. Indeed,

$$\tilde{m} + 0 = \tilde{m}, \text{ but } \tilde{m} + (-\tilde{m}) \neq 0$$

$$\tilde{m} \cdot 1 = \tilde{m}, \text{ but } \tilde{m} \cdot (\tilde{m}^{-1}) \neq 1$$

It means that ($\tilde{a}\tilde{x}=\tilde{b}$) is not equivalent to ($\tilde{x}=\tilde{b}/\tilde{a}$). It may be shown that ($\tilde{x}=\tilde{b}/\tilde{a}$) $\Rightarrow$ ($\tilde{a}\tilde{x} \supseteq \tilde{b}$) i.e. using usual methods we extend the set of solutions of the equations. For example, the solution of the equation $\tilde{a}\tilde{x}=\tilde{b}$ may not exist, but the formal application of usual methods (if $\tilde{a}\neq\tilde{0}$) will give $\tilde{x}=\tilde{b}/\tilde{a}$.

It follows that in order to solve (9) we have to look for algorithms other than usual ones.

As it was mentioned above we will deal only with LR-type fuzzy numbers (L, R being arbitrary admissible functions but the same for all coefficients and right-hand sides).

Let $\tilde{a}_{ij}=(a_{ij},\alpha_{ij},\beta_{ij})_{LR}$, $\tilde{b}_i=(b_i,\underline{b}_i,\overline{b}_i)_{LR}$, $0\notin \text{supp}\ \tilde{a}_{ij}$, $0\notin \text{supp}\ \tilde{b}_i$

Since the set of LR-type fuzzy numbers is closed under arithmetic operations (having in mind approximate formulae for multiplication and division), it is natural to suppose that $\tilde{x}_i$ are also in this set i.e. we obtain $\tilde{x}_i$ as

$$\tilde{x}_i=(x_i,p_i,q_i)_{LR},\ p_i\geq 0,\ q_i\geq 0,\ 0\notin \text{supp}\ \tilde{x}_i$$

Some difficulties may arise. For example, consider an LR-type fuzzy linear system of two equations (i.e. n=2) :

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 = \tilde{b}_1 \\ \tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 = \tilde{b}_2 \end{cases} \qquad (11)$$

where $\tilde{a}_{11}>0$, $\tilde{a}_{12}<0$.

Suppose that we apriori know that $\tilde{x}_1>0$, $\tilde{x}_2>0$. Then

$$\tilde{a}_{11}\tilde{x}_1 = (a_{11}x_1,\alpha_{11}x_1 + p_1a_{11},\beta_{11}x_1 + q_1a_{11})_{LR}$$

$$\tilde{a}_{12}\tilde{x}_2 = (a_{12}x_2,\alpha_{12}x_2 - q_2a_{12},\beta_{12}x_2 - p_2a_{12})_{RL}$$

The problem is that the sum of LR-type fuzzy number and

RL-type one is undefiened.

This fact leads us to the necessity of consideration only LL-type fuzzy numbers. Let us notice that the generality of our problem will not be affected very much (by variating the type of L and values of respective parametrs we can approximate a wide set of membership functions) but at the same an important result will be obtained, namely, the common (that is independent of signs of factors) formula for multiplication of LL-type fuzzy numbers.

Lemma. Let $\tilde{a}=(a,\alpha,\beta)_{LL}$; $\tilde{b}=(b,\varepsilon,\gamma)_{LL}$; $\tilde{c}=(c,c_1,c_2)_{LL}=\tilde{a}\cdot\tilde{b}$,

$0\notin \operatorname{supp}\tilde{a}$, $0\notin \operatorname{supp}\tilde{b}$

Then

1) $c^{*}=c_1+c_2=|b|(\alpha+\beta)+|a|(\varepsilon+\gamma)$

2) $c^{**}=c_1-c_2=b(\alpha-\beta)+a(\varepsilon-\gamma)$

The proof follows from (5a)-(5d) if we set L=R.

According to this lemma :

$$c_1=\frac{1}{2}(c^{*}+c^{**}),\quad c_2=\frac{1}{2}(c^{*}-c^{**})$$

Thus, $$\tilde{a}\cdot\tilde{b}=(ab,\tfrac{1}{2}(c^{*}+c^{**}),\ \tfrac{1}{2}(c^{*}-c^{**}))_{LL} \qquad (12)$$

We now turn back to the solution of fuzzy linear system. Consider the i-th equation of system (9) :

$$\sum_{j=1}^{n}\tilde{a}_{ij}\tilde{x}_j=\tilde{b}_i \qquad (13)$$

The left-hand side of (13) is a LL-type fuzzy number $\tilde{A}_i=\sum_{j=1}^{n}\tilde{a}_{ij}\tilde{x}_j$ or $\tilde{A}_i=(A_i,\underline{A}_i,\overline{A}_i)_{LL}$.

For example, for system (11) :

$A_1 = a_{11}x_1 + a_{12}x_2$

$\underline{A}_1 = \alpha_{11}x_1 + p_1 a_{11} + \alpha_{12}x_2 - q_2 a_{12}$; $\overline{A}_1 = \beta_{11}x_1 + q_1 a_{11} + \beta_{12}x_2 - p_2 a_{12}$

The right-hand side of (13) is also a LL-type fuzzy number $\tilde{b}_i = (b_i, \underline{b}_i, \overline{b}_i)_{LL}$. Two fuzzy numbers are equal iff their membership functions coincide. For LR-type fuzzy numbers (and also, of course, for LL-type ones) this means that values of 3 parameters, which characterize such fuzzy numbers must be equal. Consequently, this leads to the system of ordinary (non-fuzzy) equations :

$$\begin{cases} A_i = b_i \\ \underline{A}_i = \underline{b}_i \\ \overline{A}_i = \overline{b}_i \end{cases}, \quad i = \overline{1,n} \qquad (14)$$

with the following restrictions :

$$p_j \geqslant 0, \; q_j \geqslant 0 \qquad (15)$$

$$, \; j = \overline{1,n}$$

$$0 \notin \operatorname{supp} \tilde{x}_j \qquad (16)$$

The system (14) is a linear system of 3n equations with 3n variables - x_j, p_j, q_j $(j=\overline{1,n})$. If the system (14) has unique solution which also satisfies the restrictions (15), (16), then our problem is solved. In opposite case the problem (14)-(16) can not be solved. So in general case the exact solution of system (9) may not exist and it makes sence to change the notion of such solution and to look for approximate one (quasi-solution). According to [5] let us consider the following problem :

$$\begin{cases} \min\limits_{x_j, p_j, q_j} F(A_1 - b_1, \dots, A_n - b_n, \underline{A}_1 - \underline{b}_1, \dots, \underline{A}_n - \underline{b}_n, \dots, \overline{A}_n - \overline{b}_n) \\ p_j \geqslant 0 \qquad\qquad j = \overline{1,n} \\ x_j \geqslant 0 \\ 0 \notin \operatorname{supp} x_j \end{cases} \tag{17}$$

where F is a certain functional measuring the deviation of the lefthand side of (14) from the right-hand one. This allows also to solve systems in which the number of equtions (m) is more than the number of variables (n) (to solve in the sence of minimization of functional F).

In the simplest case when F is a quadratic functional (17) is transformed to :

$$\begin{cases} \min\limits_{x_j, p_j, q_j} \sum\limits_{i=1}^{m} \left[k_1 (A_i - b_i)^2 + k_2 (\underline{A}_i - \underline{b}_i)^2 + k_3 (\overline{A}_i - \overline{b}_i)^2 \right] \\ p_j \geqslant 0 \qquad i = \overline{1,m} \qquad (m \geqslant n) \\ q_j \geqslant 0 \qquad j = \overline{1,n} \\ 0 \notin \operatorname{supp} x_j \end{cases} \tag{18}$$

where k_1, k_2, k_3 are "weight" coefficients.

According to the arithmetical operations on fuzzy numbers introduced above, we have :

$$A_i = \sum_{j=1}^{n} a_{ij} x_j \ ; \ \underline{A}_i = \frac{1}{2}(A_i^{*} + A_i^{**}); \ \overline{A}_i = \frac{1}{2}(A_i^{*} - A_i^{**}) \tag{19}$$

where $A_i^{*} = \underline{A}_i + \overline{A}_i = \sum\limits_{j=1}^{n} \left(|x_j| (\alpha_{ij} + \beta_{ij}) + |a_{ij}| (p_j + q_j) \right)$

$$A_i^{**} = \underline{A}_i - \overline{A}_i = \sum_{j=1}^{n} \left(x_j (\alpha_{ij} - \beta_{ij}) + a_{ij} (p_j - q_j) \right) \tag{20}$$

It makes sence to consider the following problem instead of (17) :

$$\begin{cases} \min\limits_{x_j,p_j,q_j} F(A_1-b_1,\dots,A_m-b_m,A_1^*-(\underline{b}_1+\overline{b}_1),\dots,A_m^*-(\underline{b}_m+\overline{b}_m),A_1^{**}-(\underline{b}_1-\overline{b}_1),\dots,A_m^{**}-(\underline{b}_m-\overline{b}_m)) \\ p_j \geqslant 0 \qquad j=\overline{1,n} \\ q_j \geqslant 0 \qquad i=\overline{1,m} \qquad (m \geqslant n) \\ 0 \notin \operatorname{supp} \tilde{x}_j \end{cases} \tag{21}$$

Let us discuss the restriction $0 \notin \operatorname{supp} \tilde{x}_j$. It is clear that $\operatorname{supp} \tilde{x}_j = [x_j - L_+^{-1}(0)p_j;\ x_j + L_+^{-1}(0)q_j]$, where $L_+^{-1}(0)$ was introduced above. For example, for $L(x) = \max(0; 1-|x|^p)$, $(p \geqslant 0)$ $L_+^{-1}(0) = 1$, $\operatorname{supp} \tilde{x}_j = [x_j - p_j; x_j + q_j]$.

The restriction $0 \notin \operatorname{supp} \tilde{x}_j$ is equivalent to the following one :

$$(x_j - L_+^{-1}(0)p_j)(x_j + L_+^{-1}(0)q_j) \geqslant 0$$

Thus, the problem of solution of LL-type fuzzy linear system is reduced to an ordinary (non-fuzzy)non-linear optimization problem :

$$\begin{cases} \min\limits_{x_j,p_j,q_j} F(A_1-b_1,\dots,A_m-b_m,A_1^*-(\underline{b}_1+\overline{b}_1),\dots,A_m^*-(\underline{b}_m+\overline{b}_m),A_1^{**}-(\underline{b}_1-\overline{b}_1),\dots,A_m^{**}-(\underline{b}_m-\overline{b}_m)) \\ (x_j - L_+^{-1}(0)p_j)(x_j + L_+^{-1}(0)q_j) \geqslant 0 \\ p_j \geqslant 0 \qquad i=\overline{1,m} \qquad (m \geqslant n) \\ q_j \geqslant 0 \qquad j=\overline{1,n} \end{cases} \tag{22}$$

where A_i, A_i^*, A_i^{**} $(i=\overline{1,m})$ are given by (19), (20).

The unique solution of (22) exists, for example, if the functional F is the quadratic one.

Often, especially in practice, we may have some apriori information about signs of $\tilde{x}_j$. This happens, as a rule, in regression's problems, when the type of dependance is apriori known and it is necessary only to find the values of respective parametres. Besides, the signs of $\tilde{x}_j$ may be often determined in practice from the physical nature of the problem. This apriori information about signs of $\tilde{x}_j$ is very helpful and it makes the expressions for the functional and restrictions of (22) much simpler. When signs of all $\tilde{x}_j$ are known, the restrictions are reduced to linear ones.

Now let us mention one important particular case of LR-type (or LL-type) fuzzy linear system.

Let membership functions of $\tilde{a}_{ij}$ and $\tilde{b}_i$ be symmetric ones, i.e. $\tilde{a}_{ij} = (a_{ij}, \alpha_{ij}, \alpha_{ij})_{LL}$, $\tilde{b}_i = (b_i, \hat{b}_i, \hat{b}_i)_{LL}$. This case is common in practice. Such fuzzy numbers will be called symmetric fuzzy numbers. For such numbers the above lemma certainly holds. We have :

$$\tilde{a} \cdot \tilde{b} = (ab, \alpha|b| + \varepsilon|a|, \alpha|b| + \varepsilon|a|)_{LL}$$

where $\tilde{a} = (a, \alpha, \alpha)_{LL}$; $\tilde{b} = (b, \varepsilon, \varepsilon)_{LL}$

Also notice the fact that

$$\tilde{a} + \tilde{b} = (a+b, \alpha+\varepsilon, \alpha+\varepsilon)_{LL}$$

Thus, the set of symmetric LR-type fuzzy numbers is also closed under operations of addition and multiplication (having in mind again the approximate formula for multiplication). So we will look for $\tilde{x}_j$ belonging to this set, i.e. $\tilde{x}_j = (x_j, p_j, p_j)_{LL}$

Then :

$$A_i = \sum_{j=1}^{n} a_{ij}x_j \; ; \; \underline{A}_i = \overline{A}_i = \hat{A}_i = \sum_{j=1}^{n} (|a_{ij}| p_j + |x_j| \alpha_{ij}) \qquad (23)$$

(17) is transformed to :

$$\begin{cases} \min\limits_{x_j, p_j} F(A_1 - b_1, \dots, A_m - b_m, \hat{A}_1 - \hat{b}_1, \dots, \hat{A}_m - \hat{b}_m) \\ |x_j| - L_+^{-1}(0) p_j \geq 0 \qquad j = \overline{1,n} \; (m \geq n) \\ p_j \geq 0 \end{cases} \qquad (24)$$

(18) is respectivly transforms to :

$$\begin{cases} \min\limits_{x_j, p_j} \sum_{i=1}^{n} \left[k_1 \left(\sum_{j=1}^{n} a_{ij}x_j - b_i \right)^2 + k_2 \left(\sum_{j=1}^{n} (|a_{ij}| p_j + |x_j| \alpha_{ij}) - \hat{b}_i \right)^2 \right] \\ |x_j| - L_+^{-1}(0) p_j \geq 0 \qquad i = \overline{1,m} \\ p_j \geq 0 \qquad j = \overline{1,n} \qquad (m \geq n) \end{cases} \qquad (25)$$

In conclusion we remark that given a fuzzy system it is sometimes useful to solve it at first as a non-fuzzy one, neglecting the fuzziness of coefficients. This is for two reasons :

1) by solving a respective non-fuzzy system we may get some apriori information about signs of $\tilde{x}_j$;
2) the solution of the non-fuzzy system may be used as an initial point for non-linear optimization problem (22) or (24).

Numerical example.

Consider the following system (n=2, m=2) :

$$\begin{cases} \tilde{a}_{11}\tilde{x}_1 + \tilde{a}_{12}\tilde{x}_2 = \tilde{b}_1 \\ \tilde{a}_{21}\tilde{x}_1 + \tilde{a}_{22}\tilde{x}_2 = \tilde{b}_2 \end{cases} \qquad (26)$$

where

$\tilde{a}_{11}=(8;\ 0.05;\ 0.05)_{LL}$; $\tilde{a}_{12}=(3;\ 0.01;\ 0.01)_{LL}$

$\tilde{a}_{21}=(3;\ 0.02;\ 0.02)_{LL}$; $\tilde{a}_{22}=(4;\ 0.04;\ 0.04)_{LL}$

$\tilde{b}_1=(31;\ 1.;\ 1.)_{LL}$; $\tilde{b}_2=(26;\ 0.8;\ 0.8)_{LL}$

The respective non-fuzzy system is :

$$\begin{cases} 8\,x_1 + 3\,x_2 = 31 \\ 3\,x_1 + 4\,x_2 = 26 \end{cases} \qquad (27)$$

The solution is : $x_1 = 2$; $x_2 = 5$.

Using this fact we may obtain that $\tilde{x}_1 > 0$, $\tilde{x}_2 > 0$.

Now reduce the problem (26) to a non-linear optimization problem using methods described in this article. We have (in case $k_1 = k_2 = 1$) :

$$\begin{cases} \min\limits_{x_1, x_2, p_1, p_2} \Big\{ (8x_1+3x_2-31)^2+(8p_1+0{,}05|x_1|+3p_2+0{,}01|x_2|-1)^2+ \\ \quad +(3x_1+4x_2-26)^2+(3p_1+0{,}02|x_1|+4p_2+0{,}04|x_2|-0{,}8)^2 \Big\} \\ |x_1|-p_1 \geqslant 0 \\ |x_2|-p_2 \geqslant 0 \\ p_1 \geqslant 0 \\ p_2 \geqslant 0 \end{cases} \qquad (28)$$

(recall that $L_+^{-1}(0)=1$)

Taking into account the information about signs of $\tilde{x}_1$ and $\tilde{x}_2$ we obtain the quadratic programming problem. Its solution is :

$$x_1 = 2 \qquad p_1 = 0.07$$
$$x_2 = 5 \qquad p_2 = 0.08$$

Thus, $\tilde{x}_1=(2;\ 0.07;\ 0.07)_{LL}$; $\tilde{x}_2=(5;\ 0.08;\ 0.08)_{LL}$.

If we choose p=0.1 (we use small values of p in order to apply (7) to illustrate the process) the errors of application of approximate formulae for multiplication will be :

$\varepsilon_{11} \approx 5 \cdot 10^{-3}$ $\quad \varepsilon_{12} \approx 3 \cdot 10^{-3}$

$\delta_{11} \approx 0{,}013$ $\quad \delta_{12} \approx 0{,}014$ $\quad \delta_1 = \delta_{11} + \delta_{12} \approx 0{,}027$

$\varepsilon_{21} \approx 6 \cdot 10^{-3}$ $\quad \varepsilon_{22} \approx 6 \cdot 10^{-3}$

$\delta_{21} \approx 0{,}014$ $\quad \delta_{22} \approx 0{,}024$ $\quad \delta_2 = \delta_{21} + \delta_{22} \approx 0{,}037$

Even these rough estimates show that in this case we have correctly used the approximation formulae.

For comparison let us solve the system (26) by Kramer's method. We will have :

$x_1 = 2 \qquad p_1 = 0.4$

$x_2 = 5 \qquad p_2 = 0.6$

i.e. $\tilde{x}_1 = (2;\ 0.4;\ 0.4)_{LL}$; $\tilde{x}_2 = (5;\ 0.6;\ 0.6)_{LL}$.

We observe that the "fuzziness" of the solution increases significantly by the reasons mentioned in this article.

Reserences

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