SOME PROPERTIES OF POSSIBILISTIC LINEAR EQUALITY AND INEQUALITY SYSTEMS

R.Fuller
Computing Center of the ELTE University, H-1502,
Budapest 112, pf 157,Hungary

Linear equality and inequality systems with fuzzy parameters defined by the extension principle are called *possibilistic linear equality and inequality systems.* The solution of these systems is defined in the sense of Bellman-Zadeh [1]. This paper investigate stability and continuity properties of the solution in the above-named systems.

Keywords: Extension principle, stability

1.Preliminaries.

Definition 1. A fuzzy number is a fuzzy set $\tilde{a}$, $\tilde{a} : \mathbf{R} \to [0,1] = I$, which is upper-semicontinuous, normal and convex, i.e. (i) $[\tilde{a}]^{\alpha} = \{x \mid \tilde{a}(x) \geqq \alpha \}$ is a closed interval, (ii) $\exists x$ such that $\tilde{a}(x) = 1$, (iii) $\tilde{a}(\lambda x + (1-\lambda)y) \geqq \tilde{a}(x) \wedge \tilde{a}(y)$, for $\lambda \in I$.

By $\mathcal{F}$ we denote the set of all fuzzy numbers $\tilde{a}$ with the membership function having the following properties

(i) $\tilde{a}(t) = 0$, outside of some interval $[c,d]$,
(ii) there are real numbers a and b ,$c \leqq a \leqq b \leqq d$ such that

$\tilde{a}$ is strictly increasing on the interval $[c,d]$, strictly decreasing on $[b,d]$ and $\tilde{a}(t) = 1$ for each $t \in [a,b]$.

If $\tilde{a}$, $\tilde{b} \in \mathcal{F}$ and $\lambda \in \mathbf{R}$ then $\tilde{a} + \tilde{b}$, $\tilde{a} - \tilde{b}$, $\lambda\tilde{a}$ are defined by the Zadeh's extension principle in the usual way.

Let $*$ be a natural relation defined on the real axis $\mathbf{R}$. The $*$ relation can be extended to fuzzy numbers by means of the following extension principle [5].

Definition 2. Let $\tilde{a}$, $\tilde{b}$ be fuzzy numbers. Then the truth value of the assertion $\tilde{a} * \tilde{b}$ (e.g. "$\tilde{a}$ is greater than $\tilde{b}$ ", which we write $\tilde{a} \geqq \tilde{b}$),is $\text{Poss}(\tilde{a} * \tilde{b})$ defined as

$$\text{Poss}(\tilde{a} * \tilde{b}) = \sup_{x*y} \tilde{a}(x) \wedge \tilde{b}(y)$$

It is easily checked that [3]

$$\text{Poss}(\tilde{a} * \tilde{b}) = \sup_{t*0} (\tilde{a} - \tilde{b})(t)$$

Let $L > 0$ be a real number. By $\text{Lip}(L)$ we denote the set of all functions $f\colon \mathbf{R}^k \to \mathbf{R}, (k \geqq 1)$ satisfying the Lipschitz condition (in norm $\|.\|_1$) with constant L, i.e.

$$|f(x) - f(y)| \leqq L\|x - y\|_1, \qquad \forall\ x,y \in \mathbf{R}^k$$

where $\|x\|_1 = |x_1| + \ldots + |x_k|$.

We define a metric D in $\mathcal{F}$ by the equation

$$D(\tilde{a},\tilde{b}) = \sup_{\alpha \in I} \max_{i=1,2} \{|a_i(\alpha) - b_i(\alpha)|\},$$

where $\tilde{a},\tilde{b} \in \mathcal{F}$ and $[\tilde{a}]^\alpha = [a_1(\alpha), a_2(\alpha)]$, $[\tilde{b}]^\alpha = [b_1(\alpha), b_2(\alpha)]$.

A symmetrical triangular fuzzy number $\tilde{a}$ denoted by $\tilde{a} = (a,\alpha)$ is defined as

$$\tilde{a}(t) = \begin{cases} 1 - |a-t|/\alpha & \text{if } |a-t| \leqq \alpha, \\ 0 & \text{otherwise} \end{cases}$$

where $a \in \mathbf{R}$ is the center and $\alpha > 0$ is the width of $\tilde{a}$.

2. Results

Consider the following possibilistic linear equality and inequality system:

$$\tilde{a}_{i1}x_1 + \dots + \tilde{a}_{in}x_n * \tilde{b}_i \ , \ i = 1,\dots,m \qquad (1)$$

where $x = (x_1,\dots x_n)$ is a vector of real variables, $\tilde{a}_{ij}$, $\tilde{b}_i \in \mathcal{F}$ are fuzzy numbers and $*$ denotes $<$, $\leqq$, $=$, $\geqq$,or $>$.
We can define the solution of the system (1) as

$$\mu(x) = \min_i \mathrm{Poss}(\tilde{a}_{i1}x_1 + \dots + \tilde{a}_{in}x_n * \tilde{b}_i) \ , \ \forall x \in \mathbf{R}^n, \qquad (2)$$

or, equivalently

$$\mu(x) = \min_i \sup_{a_{ij},b_i:\ \sum_{j=1}^{n} a_{ij}x_j \ * b_i} \tilde{a}_{i1}(a_{i1}) \wedge \dots \wedge \tilde{a}_{in}(a_{in}) \wedge \tilde{b}_i(b_i).$$

A measure of consistency for the system (1) is [2]

$$\mu^* = \mu(x^*) = \max_{x \in \mathbf{R}^n} \mu(x);$$

and x^* is the maximizing (or best) solution.
In the next theorem we see that from Lipschitz-continuity of fuzzy parameters in (1) follows the Lipschitz-continuity of the solution (2).

Theorem 1. Let $\tilde{a}_{ij}$, $\tilde{b}_i \in \mathcal{F} \cap \mathrm{Lip}(L)$ be fuzzy numbers. Then

$$\mu \in \mathrm{Lip}(LS)$$

where $S = \max_{i,j} S_{ij}$, $S_{ij} = \max\{ \ |t| \ : \ t \in \mathrm{supp}\ \tilde{a}_{ij}\}$.
The proof of this theorem is based on the following lemmas:
(I am ready to send the complete proof of this theorem for anyone interested)

Lemma 1. Let $\tilde{a}, \tilde{b}, \tilde{c} \in \mathcal{F}$ be fuzzy numbers and let $\lambda, \mu \in \mathbf{R}$ be real numbers. Then

$$D(\lambda\tilde{a},\mu\tilde{a}) \leqq |\lambda - \mu|\max\{|t| : t \in \operatorname{supp}\tilde{a}\}$$

$$D(\tilde{a} - \tilde{b}, \tilde{c} - \tilde{b}) = D(\tilde{a}, \tilde{c})$$

Lemma 2. Let $\tilde{a}_{ij}, \tilde{b}_i \in \mathcal{F} \cap \mathrm{Lip}(L)$ and $x = (x_1,\ldots x_n) \in \mathbf{R}^n$. Then

$$\tilde{a}_{i1}x_1 + \ldots + \tilde{a}_{in}x_n - \tilde{b}_i \in \mathcal{F} \cap \mathrm{Lip}\left(\frac{L}{\|x\|_1 + 1}\right).$$

Lemma 3. Let $\tilde{a}, \tilde{b} \in \mathcal{F} \cap \mathrm{Lip}(L)$ and $\delta \geqq 0$. If $D(\tilde{a}, \tilde{b}) \leqq \delta$, then

$$\sup_{t\in\mathbf{R}} |\tilde{a}(t) - \tilde{b}(t)| \leqq L\delta$$

In many important cases instead of exact fuzzy numbers $\tilde{a}_{ij}$, $\tilde{b}_i$ in (1) only their approximations $\tilde{a}^{\delta}_{ij}$, $\tilde{b}^{\delta}_i$ are known [4],such that

$$D(\tilde{a}_{ij},\tilde{a}^{\delta}_{ij}) \leqq \delta , \quad D(\tilde{b}_i,\tilde{b}^{\delta}_i) \leqq \delta , \tag{3}$$

Then we get the following so-called perturbed system

$$\tilde{a}^{\delta}_{i1}x_1 +\ldots+ \tilde{a}^{\delta}_{ij}x_n * \tilde{b}^{\delta}_i , \; i=1,\ldots,m \tag{4}$$

We denote the solution,the measure of consistency and the maximizing solution of the system (4) by $\mu^{\delta}, \mu^*(\delta)$ and $x^*(\delta)$ respectively.

The following theorem shows that a small perturbation (in metric D) of the fuzzy parameters $\tilde{a}_{ij}, \tilde{b}_i \in \mathcal{F} \cap \mathrm{Lip}(L)$ may cause only a small deviation in the solution.

Theorem *2. Let* $\tilde{a}_{ij}, \tilde{b}_i, \tilde{a}^{\delta}_{ij}, \tilde{b}^{\delta}_i \in \mathcal{F} \cap \mathrm{Lip}(L)$. If (3) holds, then

$$\|\mu - \mu^{\delta}\|_C = \sup_{x\in\mathbf{R}^n} |\mu(x) - \mu^{\delta}(x)| \leqq L\delta$$

$$|\mu^* - \mu^*(\delta)| \leqq L\delta .$$

The proof of this theorem is based on the above-mentioned lemmas.

QUESTION 1. Does continuity of fuzzy numbers in system (1) imply the continuity of solution μ ?

QUESTION 2. Will the solution μ be continuous in the case of non compactely-supported Lipschitzian fuzzy numbers ?

QUESTION 3. Does the solution μ have the stability property if the fuzzy parameters are continuous ?

QUESTION 4. Will the maximizing solution x^* be stable under small variations (in metric C) in the membership functions of the Lipschitzian fuzzy numbers ?

3. Numerical example.

Consider the following classical systems of linear inequalities

original system	perturbed system	
$-x_1-x_2 \leqq -4$	$-x_1-x_2 \leqq -4$	
$x_1-x_2 \leqq 0$	$(1+\delta)x_1-x_2 \leqq 0$	(5)
$-x_1+x_2 \leqq 0$	$-(1-\delta)x_1+x_2 \leqq 0$	
$0 \leqq x_1,x_2 \leqq 2$	$0 \leqq x_1,x_2 \leqq 2$	

The solution of the original system is $x_1^{opt} = x_2^{opt} = 2$ and the perturbed system has no solution for every $\delta > 0$.

Consider now the above systems with triangular fuzzy numbers

original system	perturbed system
$-(1,\alpha)x_1 - (1,\alpha)x_2 \leqq -(4,\alpha)$	$-(1,\alpha)x_1 - (1,\alpha)x_2 \leqq -(4,\alpha)$
$(1,\alpha)x_1 - (1,\alpha)x_2 \leqq (0,\alpha)$	$(1+\delta,\alpha)x_1-(1,\alpha)x_2 \leqq (0,\alpha)$
$-(1,\alpha)x_1 + (1,\alpha)x_2 \leqq (0,\alpha)$	$-(1-\delta,\alpha)x_1+(1,\alpha)x_2 \leqq (0,\alpha)$
$0 \leqq x_1,x_2 \leqq 2$	$0 \leqq x_1,x_2 \leqq 2$

Using the above notations we get

$$\mu^* = 1, \qquad \mu^*(\delta) = 1 - \frac{\delta}{\alpha} \cdot \frac{4}{10+\delta},$$

$$x_1^* = x_2^* = 2, \qquad x_1^*(\delta) = x_2^*(\delta) = \frac{4}{2+\delta},$$

so

$$x_1^*(\delta) \to x_1^* = x_1^{opt}, \quad x_2^*(\delta) \to x_2^* = x_2^{opt} \quad \text{as } \delta \to 0. \tag{6}$$

Remark. From (6) it follows that the *fuzzification* of the classical problem (5) can be considered as a "regularized" formulation of the classical problem (5) /i.e. we get a so-called self-reguralization method for solving the instable problem (5)/.

References

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