SOME PROPERTIES OF FUZZY POLYNOMIALS

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Summary The stability of the fuzzy solution of a polynomial equation and an algebraic property of the solution are investigated. We can prove the stability of the solution. The other problem to be studied is that under which assumptions we can guarantee that the fuzzy solution of a polynomial equation of degree n consists of the union of at most n fuzzy numbers.

Keywords: fuzzy polynomial, fuzzy number

1. Complex fuzzy numbers

Let $\mathbb{R}$ and $\mathbb{C}$ denote the field of real and complex numbers respectively. Let L be given continuous function from $\mathbb{C}$ into the closed interval [0,1] with the properties

(i) $L(0) = 1$ and $L(z) < 1 \qquad z \in \mathbb{C}\setminus\{0\}$

(ii) $\exists \quad \varphi : \mathbb{R}_0^+ \to [0,1] \longrightarrow$

$L(z) = \varphi(|z|) \qquad z \in \mathbb{C}$ where

$\mathbb{R}_0^+ = \{x \in \mathbb{R} | x \geq 0\}$

(iii) φ is decreasing

(iiii) There exists a constant $c_0 > 0$ so that for each $z, w \in \mathbb{C}$

$$|L(z) - L(w)| \leq c_0 |z - w|$$

Definition 1.1 A complex fuzzy set $\tilde{z} : \mathbb{C} \to [0,1]$ is said to be an L type complex fuzzy number iff there exist $z \in \mathbb{C}$ $\alpha \in \mathbb{R}^+$ and L is a function having the properties we mentioned above so that

$$\tilde{z}(t) = L\left(\frac{z-t}{\alpha}\right).$$

Then z and α is called the mean and spead of $\tilde{z}$. Symbolically we write $\tilde{z} = (z,\alpha)_L$

For example $$L(t) := \begin{cases} 1-|t| & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| > 1 \end{cases} \qquad t \in \mathbb{C}$$

In that case the L type complex fuzzy numbers are

$$\tilde{z}(t) = \begin{cases} 1-\frac{|z-t|}{\alpha} & \text{if } |z-t| \leq 1 \\ 0 & \text{if } |z-t| > 1 \end{cases}$$

where $z \in \mathbb{C}$ and $\alpha \in R^+$

Now let us define the addition and scalar multiplication of two fuzzy numbers in the following way (see [1])

Definition 1.2 $\tilde{z} = (z,\alpha)_L$, $\tilde{w} = (w,\beta)_L$ $\lambda \in \mathbb{C}$

$$\tilde{z} + \tilde{w} := (z + w, \alpha + \beta)_L$$

$$\lambda\tilde{z} := (\lambda z , |\lambda| \alpha)_L$$

So we can define a complex fuzzy polynomial $\tilde{P}_n$ of degree n. Let us consider a complex polynomial P_n.

$$P_n(z) = \sum_{i=0}^{n} a_i z^i \quad \text{where} \quad a_i \in \mathbb{C} \quad i = 0,\dots,n \quad z \in \mathbb{C}$$
$$a_n \neq 0$$

Definition 1.3 The fuzzy polynomial $\tilde{P}_n$ is defined as below

$$\tilde{P}_n(z) := \sum_{i=0}^{n} \tilde{a}_i z^i \quad \text{where} \quad \tilde{a}_i = (a_i, \alpha)_L \text{ are complex}$$

fuzzy numbers with equal spreads for the sake of the simplicity.

If z is a complex number $\tilde{P}_n(z)$ will be a complex fuzzy number.

We can write the explicit form of $\tilde{P}_n$

$$\tilde{P}_n(z) = \left(P_n(z),\ \alpha\gamma(z)\right)_L \qquad \text{where}$$

$$\gamma(z) = \sum_{i=0}^{n} |z|^i$$

Definition 1.4 Let $\tilde{z} = (z,\alpha)_L$ and $\tilde{w} = (w,\beta)_L$ be complex fuzzy numbers. Then let $v(\tilde{z}=\tilde{w})_L$ be the degree of satisfaction of the assertion $\tilde{z} = \tilde{w}$, where

$$v(\tilde{z} = \tilde{w})_L = L\left(\frac{z - w}{\alpha + \beta}\right)$$

Now we can define the fuzzy solution of the equation $\tilde{P}_n(z) = \tilde{0}$ where $\tilde{0} = (0,\beta)_L$ is a complex fuzzy number.

Definition 1.5 The fuzzy set $\tilde{z}^*(t) = v\left(\tilde{P}_n(t) = \tilde{0}\right)$ is said to be the solution of the equation $\tilde{P}_n(z) = \tilde{0}$ that is

$$\tilde{z}^*(t) = L\left(\frac{P_n(t)}{\alpha\gamma(t) + \beta}\right)$$

As we can see the solution $\tilde{z}^*$ is really a fuzzy set. We are interesting in two problems

(i) Is the solution $\tilde{z}^*$ stabil or not with respect to changing the centers of the fuzzy numbers?

(ii) Under which assumptions will it be true that the fuzzy set $\tilde{z}^*$ is the union of at most n fuzzy numbers?

The problem (i) is solved by R.Fuller for Fuzzy Linear Programming Problem. (see [2])

2. Stability

For $\delta > 0$, let $a_i \in \mathbb{C}$ be complex numbers such that

$$\max_{0 \leq i \leq n} |a_i - a_i^{\delta}| \leq \delta$$

Then we consider the complex fuzzy numbers

$$\tilde{a}_i^{\delta} = \left(a_i^{\delta}, \alpha\right)_L$$

with perturbed centers and the fuzzy polynomial

$$\tilde{P}_n^{\delta}(z) = \sum_{i=0}^{n} \tilde{a}_i^{\delta} z^i \qquad z \in \mathbb{C}$$

Then the solution of the fuzzy equation $\tilde{P}_{\delta}(z) = \tilde{0}$ is

$$\tilde{z}_{\delta}^{*}(t) = L\left(\frac{P_n^{\delta}(t)}{\alpha\gamma(t)+\beta}\right)$$

What can we say about the quantity

$$\|\tilde{z}^{*} - \tilde{z}_{\delta}^{*}\|_{\infty} = \sup_{t \in \mathbb{C}} |\tilde{z}^{*}(t) - \tilde{z}_{\delta}^{*}(t)|$$

Theorem 2.1

For each $\delta > 0$,

$$\|\tilde{z}^{*} - \tilde{z}_{\delta}^{*}\|_{\infty} \leq c_0 \frac{\delta}{\alpha}$$

where α is the common spread of the complex fuzzy numbers $\tilde{a}_i$ and $\tilde{a}_i^\delta$ $i = 0,1,2,\dots,n$, c_0 is the Lipschitz constant of L.

Proof. For each $t \in \mathbb{C}$,

$$\left|\tilde{z}^*(t) - \tilde{z}_\delta(t)\right| = \left|L\left(\frac{P_n(t)}{\alpha\gamma(t)+\beta}\right) - L\left(\frac{P_n(t)}{\alpha\gamma(t)+\beta}\right)\right| \leqq$$

$$\leqq c_0\left|\frac{|P_n(t)|}{\alpha\gamma(t)+\beta} - \frac{|P_n^\delta(t)|}{\alpha\gamma(t)+\beta}\right| \leqq c_0\left|\frac{P_n(t)-P_n^\delta(t)}{\alpha\gamma(t)+\beta}\right| \leqq$$

$$\leqq c_0\frac{\delta\gamma(t)}{\alpha\gamma(t)+\beta} < c_0\frac{\delta}{\alpha} \quad .$$

So we have

$$\|\tilde{z}^* - \tilde{z}^*_\delta\|_\infty \leqq c_0\frac{\delta}{\alpha}$$

This means that $\tilde{z}^*_\delta$ converges to $\tilde{z}^*$ in the uniform topology.

Very often the support of the solution $\tilde{z}^*$ is unbounded, namely it is $\mathbb{C}$. We would like to change the solution $\tilde{z}^*$ so that it has bounded support and its stability property remains valid. Moreover that solution will be the union of at most n fuzzy numbers.

Let us modify the addition and the scalar multiplication so that for $\tilde{z} = (z,\alpha)_L$ and $\tilde{w} = (w,\beta)_L$, let

$$\tilde{z} + \tilde{w} := (z + w,\ \max\{\alpha,\beta\})_L$$
$$\lambda\tilde{z} := (\lambda z,\alpha)_L$$

Now the polynomial $\tilde{P}_n(z)$ that we get is

$$\tilde{P}_n(z) = \left[P_n(z),\alpha\right]_L$$

and the solution of the equation $\tilde{P}_n(z) = \tilde{0}$ will be

$$\tilde{z}^*(t) = L\left(\frac{P_n(t)}{\alpha+\beta}\right)$$

This solution will also remain stabil with a modification.

<u>Theorem 2.2</u>

For given $\alpha,\beta \in \mathbb{R}$ and $\delta \in \mathbb{R}^+$, for each compact set G, $G \subset \mathbb{C}$ there exists a $\gamma_G \in \mathbb{R}^+$ such that

$$\sup_{t\in G} |\tilde{z}^*(t) - \tilde{z}^*_\delta(t)| \leq \frac{c_0\gamma_G}{\alpha+\beta}\,\delta$$

where c_0 is the Lipschitz constant of L.

This means that $\tilde{z}^*_\delta$ converges to $\tilde{z}^*$ in the topology of compact convergence if δ tends to zero.

<u>Proof.</u> For each $G \subset \mathbb{C}$, where G compact we have for each $t \in G$

$$|\tilde{z}^*(t) - \tilde{z}^*(t)| = \left|L\left(\frac{P_n(t)}{\alpha+\beta}\right) - L\left(\frac{P_n^\delta(t)}{\alpha+\beta}\right)\right| \leq$$

$$\leq c_0 \frac{|P_n(t) - P_n^\delta(t)|}{\alpha+\beta} \leq \frac{c_0\delta}{\alpha+\beta}\gamma(t) \leq \frac{c_0\delta}{\alpha+\beta}\cdot\gamma_G$$

where $\gamma_G = \sup_{t \in G} \gamma(t)$

3. An algebraic property of the solution $\tilde{z}^*$

Lemma 3.1 For a given polynomial P_n , there exist $\varepsilon > 0$ and G_i $i = 1,2,..,k$ $k \leq n$ closed disjoint discs such that for all distinct roots t_i $i = 1,2,...,k$ of P_n , $t_i \in G_i$ and

$$K_\varepsilon := \left\{ t \in \mathbb{C} \mid |P_n(t)| \leq \varepsilon \right\} = \bigcup_{i=1}^{n} G_i \qquad \text{hold.}$$

Proof. The polynomial P_n is continuous. Consequently $\forall\ \varepsilon > 0$ $\exists$ G_i $i = 1,2,...k$ closed circles such that for each root t_i of P_n $t_i \in G_i$ and if

$$t \in G_i \quad \text{then} \quad |P_n(t)| \leq \varepsilon \quad .$$

Let each G_i be the maximal one.
(that is if $\overline{G}_i \supset G_i$ then there exists $t \in \overline{G}_i$ so that $|P_n(t)| > \varepsilon$)
But if $\varepsilon \to 0$ diam $(G_i) \to 0$ also holds.
Taking into account that t_i $i = 1,2,...,k$ finite many distinct points we have that there exists a positive number ε such that the accompanying G_i $i = 1,2,...,k$ are disjoint circles moreover we get for that given ε

$$K_\varepsilon = \bigcup_{i=1}^{n} G_i$$

For the polynomial P_n which has the roots t_i $i = 1,2,...,k$ we define the following complex fuzzy numbers $\tilde{z}_i$ $i = 1, 2,...,k$

Let $\alpha, \beta \in \mathbb{R}^+$ $\alpha + \beta \leq \varepsilon$ be given numbers where ε is the positive number that we gave in Lemma 3.1

$$\tilde{z}_i(t) := \begin{cases} \tilde{z}^*(t) & \text{if } t \in G_i \\ 0 & \text{if } t \notin G_i \end{cases}$$

Then we get the following

Theorem 3.1

For the polynomial P_n and the numbers α, β, given as above, the solution $\tilde{z}^*$ of the equation $\tilde{P}_n(z) = \tilde{0}$ is the union of the complex fuzzy numbers $\tilde{z}_i$ $i = 1,2,...,k$, $k \leq n$ that is

$$\tilde{z}^* = \bigvee_{j=1}^{k} \tilde{z}_j$$

where $\tilde{z}_j$ are the fuzzy numbers that we defined above

Proof. Let us consider any $t \in \mathbb{C}$. Then

i., if $\exists\ i : 1 \leq i \leq k$ $t \in G_i$ where G_i as in Lemma 3.1

Then $\tilde{z}^*(t) = \tilde{z}_i(t)$ holds and

$\tilde{z}_j(t) = 0$ $j \neq i$ hence

$$\left(\bigvee_{j=1}^{k} \tilde{z}_j\right)(t) = \tilde{z}^*(t)$$

ii., if $t \notin \bigcup_{j=1}^{k} G_j \quad \Rightarrow \quad \tilde{z}_j(t) = 0 \quad j = 1,2,\dots,k$

$$\Rightarrow \left(\bigvee_{j=1}^{k} \tilde{z}_j \right)(t) = 0$$

But if $t \notin \bigcup_{j=1}^{n} G_j \quad \Leftrightarrow \quad |P_n(t)| > \varepsilon \quad \Leftrightarrow \quad \tilde{z}^*(t) = 0$

So we get for each $t \in \mathbb{C}$

$$\tilde{z}^*(t) = \left(\bigvee_{j=1}^{n} \tilde{z}_j \right)(t)$$

References

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