# ALGEBRAIC STRUCTURE OF SYMMETRICAL M - FUZZY NUMBERS

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## Abstract

In this paper we discuss the algebraic structure of symmetrical fuzzy numbers assuming that they are mappings from $R^1$ to a lattice ordered monoid, particularly to a positive or a negative cone. Using a special ordering on the set of fuzzy numbers and introducing a new extension principle of binary algebraic operations we obtain different structural properties of these fuzzy numbers.

## 1. Preliminaries

Let M be denote a *complete lattice ordered commutative monoid (c.l.o.c.m)* with zero ([1],[3]), i.e.

M1: M is a semigroup with a commutative binary operation * ;

M2: M contains an identity $e$ ;

M3: M is lattice ordered under its partial order $\leq$ with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$ , i.e. $M = [\mathbf{0}, \mathbf{1}]$;

M4: for all $a_\alpha, b \in M$ , $\alpha \in \mathcal{A}$ :

$$\left(\bigvee_{\alpha \in \mathcal{A}} a_\alpha\right) * b = \bigvee_{\alpha \in \mathcal{A}} ( a_\alpha * b ).$$

M5: the semigroup M contains a zero $\theta$.

It is easy to see, that $*$ is nondecreasing in both variables, i.e. $a_1 \leqq a_2$ and $b_1 \leqq b_2$ imply $a_1 * b_1 \leqq a_2 * b_2$. From this follows that the *negative cone*

$$[\mathbf{O}, e] = \{ a \in M : \mathbf{O} \leqq a \leqq e \}$$

is a c.l.o.c.m. with zero $\theta = \mathbf{O}$, and the *positive cone*

$$[e, \mathbf{1}] = \{ a \in M : e \leqq a \leqq \mathbf{1} \}$$

is a c.l.o.c.m. with zero $\theta = \mathbf{1}$.

We will say that the operation $*$ is a *t-norm* on M if the identity of the monoid $(M, *, e)$ is the greatest element of M, i.e. $e = \mathbf{1}$. In this case $M = [\mathbf{O}, \mathbf{1}]$ is the negative cone and $\mathbf{O} = \theta$. If we wish to emphasize that the semigroup operation is a t-norm than we will use $\top$ instead of $*$. Particularly, it is possible that $* = \top = \wedge$.

Dually, the semigroup operation $*$ will be called *t-conorm* on M if the identity of the monoid $(M, *, e)$ is the least element of M, i.e. $e = \mathbf{O}$. In this case $M = [\mathbf{O}, \mathbf{1}]$ is the positive cone and $\mathbf{1} = \theta$. If we want to refer only to t-conorms we will use the notion $\perp$ instead of $*$. Particularly it is possible that $* = \perp = \vee$. If we say that M is a *$*$-cone*, we understand that it is either a negative cone defined by a t-norm, or a positive cone defined by a t-conorm. In the sequel it will be supposed that M is a $*$-cone. If a set M possesses both cone-structures then

$$a \top b \leqq a \wedge b \leqq a \vee b$$

and

$$a \wedge b \leqq a \vee b \leqq a \perp b .$$

We remark that in the case $M = [0,1] \subset R^1$ the given t-norm and t-conorm definitions coincide with the usual ones ([5],[6]).

Let X and Y be two spaces and let be given a mapping $p : X \longrightarrow Y$. Then the triplet $(X,p,Y)$ is called a fibre bundle on Y. Here X is the fibre space, Y is the basis of the fibre bundle and $p^{-1}(y)$ is a bundle on y, where

$$p^{-1}(y) = \{x \in X : p(x) = y , y \in Y\}.$$

Note that for every $y_1, y_2 \in Y$

$$p^{-1}(y_1) \cap p^{-1}(y_2) = \emptyset$$

whenever $y_1 \neq y_2$.

## 2. The fibre bundle of symmetrical M-fuzzy numbers on $R^1$

Let $M = ([0,1],*,e,\theta,\wedge,\vee)$ be a * - cone. Introduce the following definitions:

*Definition 2.1.* A mapping $f_0 : R^1 \longrightarrow M$ will be called a *finite symmetrical M-fuzzy number on zero* if

$F_0 1$: $f_0(0) = 1$ ;

$F_0 2$: $f_0(x) = f_0(-x)$ for every $x \in R^1$ ;

$F_0 3$: $f_0(x) \geqq f_0(y)$ if $x \leqq y$, $x,y \in [0,\infty)$ ;

$F_0 4$: $\text{top } f_0 = \text{cl } \{ x \in R^1 : f_0(x) = 1 \}$ is bounded.

(Here cl denotes the closure of the set).

The set of all finite symmetrical M-fuzzy numbers on zero will be denoted by $\mathcal{F}_0$.

*Definition 2.2.* Let $a \in R^1$. A mapping $f_a : R^1 \longrightarrow M$ is called a *finite symmetrical M-fuzzy number on a* iff the mapping

$$R^1 \longrightarrow M$$
$$x \longrightarrow f_a(x+a)$$

is an element of $\mathcal{F}_0$. So the finite symmetrical M-fuzzy numbers can be represented by a pair $(f,a)$, where $f \in \mathcal{F}_0$, $a \in R^1$.

The set of all finite symmetrical M-fuzzy numbers on a will be denoted by $\mathcal{F}_a$, and the set of all finite symmetrical M-fuzzy numbers on $R^1$ will be denoted by $\mathcal{F}_R$. We have

$$\mathcal{F}_R = \bigcup_{a\in R^1} \mathcal{F}_a.$$

From the given definitions immediately follows the

*Proposition 2.3.* A mapping $f : R^1 \longrightarrow M$ belongs to $\mathcal{F}_R$ if and only if $\sup(\text{top } f) - \inf(\text{top } f) < \infty$ and there exists $a \in R^1$ such that

i) $f(a) = 1$ ;

ii) $f(a - x) = f(a + x)$ for every $x \in R^1$ ;

iii) $f(x) \geqq f(y)$ if $x \leqq y$, $x,y \in [a,\infty)$.

Here a is uniquely defined by

$$a = [\inf(\text{top } f) + \sup(\text{top } f)]/2.$$

The real number $a \in R^1$ defined in the Proposition 2.3 is called the *center* of $f \in \mathcal{F}_R$ and we write $a = \text{cent } f$.

It is easy to see that the *characteristic function of a*

$$\chi_a(x) = \begin{cases} 1, & \text{if } x = a \\ 0, & \text{if } x \neq a \end{cases}$$

belongs to $\mathcal{F}_a \subset \mathcal{F}_R$.

Let us join to $\mathcal{F}_a$ the *infinite fuzzy number* $\epsilon_a$ given by the pair $(\epsilon, a)$, where

$$\epsilon(x) = 1 \quad \text{for all } x \in R^1 .$$

Formally we say that $\epsilon_a$ is generated by $\epsilon$, cent $\epsilon_a = a$ and we distinquish $\epsilon_a$ and $\epsilon_b$ if $a \neq b$.

In the sequel if we write $f_a$ we wish to emphasize that the index a is the center of the fuzzy number created by the function f. Note these given definitions correspond to [2].

Let denote $\bar{\mathcal{F}}_a = \mathcal{F}_a \cup \{\epsilon_a\}$ and $\bar{\mathcal{F}}_R = \bigcup_{a\in R^1} \bar{\mathcal{F}}_a$. Then the following statements are valid:

*Corollary 2.4.*

i) If $a \neq b$ then $\bar{\mathcal{F}}_a \cap \bar{\mathcal{F}}_b = \emptyset$ ;

ii) If $f,g \in \bar{\mathcal{F}}_R$ and $f = g$ then cent f = cent g.

*Corollary 2.5.* $\mathcal{F}_R = (\bar{\mathcal{F}}_R, p, R^1)$ is a fibre bundle on $R^1$, where $p : \mathcal{F}_R \longrightarrow R^1$ is the projection $\mathcal{F}_R \ni f \longrightarrow \text{cent } f \in R^1$, and $\mathcal{F}_a = p^{-1}(a) \subset \mathcal{F}_R$ is the bundle on $a \in R^1$.

Let us introduce the following operations for all $f,g \in \mathcal{F}_a$:

$$(f \wedge_{F_a} g)(x) = f(x) \wedge g(x) \qquad (2.1)$$

$$(f \vee_{F_a} g)(x) = f(x) \vee g(x) \qquad (2.2)$$

$$(f *_{F_a} g)(x) = f(x) * g(x) \qquad (2.3)$$

for every $x \in R^1$.

*Proposition 2.6.* $\mathcal{F}_a = ([\chi_a,\epsilon_a], *_{F_a}, \epsilon_a, \chi_a, \wedge_{F_a}, \vee_{F_a})$ is a $*_{F_a}$ - cone.

*Proof.* From the definitions (2.1)-(2.3) follows that $\mathcal{F}_a$ is complete lattice ordered.It is easy to verify that $\chi_a$ is the least and $\epsilon_a$ is the greatest element of $\mathcal{F}_a$, $\epsilon_a$ is $\top_{F_a}$ - unity, and $\chi_a$ is $\top_{F_a}$ - zero, $\epsilon_a$ is $\perp_{F_a}$ - zero element and $\chi_a$ is $\perp_{F_a}$ - unity. Consequently, if $* = \top$ then $[\chi_a,\epsilon_a]$ is the negative cone and if $* = \perp$ then $[\chi_a,\epsilon_a]$ is the positive cone. ■

In $\mathcal{F}_R$ we define the relation $\leq_{F_R}$ as follows:

*Definition 2.7.* For every pair $f,g \in \mathcal{F}_R$ $f \leq_{F_R} g$ iff one of the following conditions is satisfied:

i) cent $f <$ cent $g$ ;

ii) cent $f =$ cent $g$ and $f(x) \leq g(x)$ for every $x \in R^1$.

Using the terminology of the corollaries 2.4. and 2.5. this definition is equivalent to the following one:

*Definition 2.7.** Let the ordering relation $\leq_{F_R}$ within each bundle $\mathcal{F}_a$, $a \in R^1$ be the same as originally given in $\mathcal{F}_a$, furthermore put $f < g$ if $f \in \mathcal{F}_a$ , $g \in \mathcal{F}_b$ and $a < b$.

*Proposition 2.8.* $(\mathcal{F}_R, \leq_{F_R})$ is lattice ordered.

*Proof.* Since $\mathcal{F}_a$ and $R^1$ are partial ordered, the relation $\leq_{F_R}$ on $\mathcal{F}_R$ is really a partial order. This ordering generates the lattice operations as follows: Let the lattice operations within each bundle $\mathcal{F}_a$ , $a \in R^1$ be given by (2.1) - (2.2), i.e. if cent f = cent g = a then

$$(f \wedge_{F_R} g)(x) = (f \wedge_{F_a} g)(x) = f(x) \wedge g(x) ,$$
$$(f \vee_{F_R} g)(x) = (f \vee_{F_a} g)(x) = f(x) \vee g(x)$$

for every $x \in R^1$, and if $f \in \mathcal{F}_a$ , $g \in \mathcal{F}_b$ and $a < b$, then

$$f \wedge_{F_R} g = f \quad \text{and} \quad f \vee_{F_R} g = g. \qquad \blacksquare$$

***Remarks 2.9.***

i) $(\mathcal{F}_R, \leq_{F_R})$ is not complete;

ii) $(\mathcal{F}_R, \leq_{F_R})$ is not linearly ordered.

iii) Let $f : [0,1] \longrightarrow M$ be a fixed monoton decreasing mapping such that $f(0) = \mathbf{1}$ , $f(1) = \mathbf{O}$. Let us consider the class $\mathcal{F}_\Delta$ of parametrical mappings

$$f_{a,d}(x) = \begin{cases} \mathbf{1}, & \text{if } x = a \text{ and } d = 0 \text{ or for all } x \text{ if } d = \infty \\ f\left[\dfrac{|x - a|}{d}\right], & \text{if } |x - a| \leq d,\ 0 < d < \infty \\ \mathbf{O}, & \text{otherwise.} \end{cases}$$

$f_{a,d} \in \mathcal{F}_\Delta$ will be called *quasi-triangular fuzzy number* ([4]) with the center a and the width d. Then $(\mathcal{F}_\Delta, \leq_{F_R})$ is a linearly ordered subsystem of $(\mathcal{F}_R, \leq_{F_R})$.

<u>3. * - extension of binary real operations to $\mathcal{F}_R$</u>

Let $\circ$ be a binary real operation on $R^1$, $*$ be a t-norm or t-conorm on M, $f, g \in \mathcal{F}_R$, and a = cent f, b = cent g.

*Definition 3.1.* The $*$ - extension $\overset{(*)}{\circ}$ of the binary operation $\circ$ to $\mathcal{F}_R$ is given by

$$(f \overset{(*)}{\circ} g)(x) = f(x + a - (a \circ b)) * g(x + b - (a \circ b)) , \quad x \in R^1. \qquad (3.1)$$

*Examples 3.2.* The $*$ - extended additive and multiplicative operations are defined by

$$(f \overset{(*)}{+} g)(x) = f(x - b) * g(x - a) ,$$

$$(f \overset{(*)}{\bullet} g)(x) = f(x + a - a{\bullet}b) * g(x + b - a{\bullet}b) .$$

*Proposition 3.3.* $\mathcal{F}_R$ is closed for the $*$ - extended binary operations, i.e. $f,g \in \mathcal{F}_R$ implies $f \overset{(*)}{\circ} g \in \mathcal{F}_R$, and $\text{cent}\ (f \overset{(*)}{\circ} g) = (\text{cent}\ f) \circ (\text{cent}\ g)$.

*Proof.* Let $f,g \in \mathcal{F}_R$ be finite symmetrical fuzzy numbers with $a = \text{cent}\ f$ , $b = \text{cent}\ g$ . It is easy to check, that the conditions i - iii) of the Proposition 2.3. are satisfied for $f \overset{(*)}{\circ} g$ and top $f \overset{(*)}{\circ} g$ is connected, so either it belongs to $\mathcal{F}_R$ or their graph is $\epsilon$.Let now $f = \epsilon_a$, $g \in \mathcal{F}_R$. Then

$$(\epsilon_a \overset{(*)}{\circ} g)(x) = g(x + b - (a \circ b)) \in \mathcal{F}_{a \circ b} \subset \mathcal{F}_R. \qquad \blacksquare$$

Corollary 3.4. If $f_a, g_b \in \mathcal{F}_R$ are generated by f and g with the center a and b , respectively, then

$$f_a \overset{(*)}{\circ} g_b = f_{a \circ b} * g_{a \circ b}.$$

*Proposition 3.5.* If $(R^1, \circ, \leqq)$ is a partial ordered commutative semigroup with the natural order $\leqq$, then $(\mathcal{F}_R, \overset{(*)}{\circ}, \leqq_{F_R})$ is a lattice ordered commutative semigroup.

*Proof.* Using that $\circ$ and $*$ are commutative and associative we can easily verify that the $*$ - extended $\overset{(*)}{\circ}$ operation on $\mathcal{F}_R$ is also commutative and associative, therefore $(\mathcal{F}_R, \overset{(*)}{\circ})$ is a commutative semigroup.

Assume now that $f \leqq_{F_R} g$ , i.e. $f \vee_{F_R} g = g$. Then

$$(f \vee_{F_R} g) \overset{(*)}{\circ} h = g \overset{(*)}{\circ} h .$$

Furthermore we know that $\text{cent}(f \overset{(*)}{\circ} h) = a \circ c$ and $\text{cent}\ (g \overset{(*)}{\circ} h) = b \circ c$ . If $a \leqq b$ then $a \circ c \leqq b \circ c$ and therefore we obtain for the right hand side that

$$(f \overset{(*)}{\circ} h) \vee_{F_R} (g \overset{(*)}{\circ} h) = (g \overset{(*)}{\circ} h).$$

If $a = b$ then $a \circ c = b \circ c$ and $f(x) \leqq g(x)$ for all $x \in R^1$. Using the monotonity of $*$ we have that

$$(f \overset{(*)}{\circ} h)(x) = f(x + a - (a \circ c)) * h(x + c - (a \circ c)) \leqq$$

$$\leqq g(x + a - (a \circ c)) * h(x + c\ (a \circ c)) = (g \overset{(*)}{\circ} h)(x). \blacksquare$$

*Proposition 3.6.* Let $(R^1, \circ, j)$ is a monoid with the neutral element $j$. Then $(\mathcal{F}_R, \overset{(\top)}{\circ}, \epsilon_j)$ is a monoid with the neutral element $\epsilon_j$, and $(\mathcal{F}_R, \overset{(\perp)}{\circ}, \chi_j)$ is a monoid with the neutral element $\chi_j$.

*Proof.* It is trivial. $\blacksquare$

*Corollary 3.7.* $(\mathcal{F}_R, \overset{(\top)}{+}, \epsilon_0)$, $(\mathcal{F}_R, \overset{(\perp)}{+}, \chi_0)$, $(\mathcal{F}_R, \overset{(\top)}{\cdot}, \epsilon_1)$ and $(\mathcal{F}_R, \overset{(\perp)}{\cdot}, \chi_1)$ are lattice ordered monoids.

## 4. Distributivity of * - extended operations

Let $A = (R^1, B)$ be an algebraic system on $R^1$, where $B$ denotes the set of the commutative and associative binary operations defined on $R^1$ and let $A_F = (\mathcal{F}_R, B^{(*)})$ be an algebraic system on $\mathcal{F}_R$, where $B^{(*)}$ is the set of the $*$ - extended binary operations of B.

*Proposition 4.1.* Let $\circ \in B$ and $\square \in B$ be distributive operations on $R^1$ and $*$ is an idempotent t-norm or t-conorm on M. Then the $*$ - extended operations $\overset{(*)}{\circ}$ and $\overset{(*)}{\square}$ on $\mathcal{F}_R$ are also distributive.

*Proof.* Let cent f = a, cent g = b and cent h = c . Then

$$[f \overset{(*)}{\circ} (g \overset{(*)}{\square} h)](x) =$$

$$= f(x+a-[a\circ(b\square c)])*(g \overset{(*)}{\square} h)(x+(b\square c)-[a\circ(b\square c)]) =$$

$$= f(x+a-[a\circ(b\square c)])*g(x+b-[a\circ(b\square c)])*h(x+c-[a\circ(b\square c)]).$$

On the other hand using that $f * f = f$ we obtain that

$$[(f \overset{(*)}{\circ} g) \overset{(*)}{\square} (f \overset{(*)}{\circ} h)](x) =$$

$$=(f \overset{(*)}{\circ} g)(x+(a\circ b)-[(a\circ b)\square(a\circ c)])*(f \overset{(*)}{\circ} h)(x+[(a\circ b)\square(a\circ c)])=$$

$$=f(x+a-[a\circ(b\square c)])*g(x+b-[a\circ(b\square c)])*f(x+a-[a\circ(b\square c)])*$$

$$*h(x+c-[a\circ(b\square c)])=$$

$$=f(x+a-[a\circ(b\square c)])*g(x+b-[a\circ(b\square c)])* h(x+c-[a\circ(b\square c)]).$$ ■

*Corollary 4.2.* $(f \overset{(*)}{+} g) \overset{(*)}{\bullet} h=(f \overset{(*)}{\bullet} h) \overset{(*)}{+} (g \overset{(*)}{\bullet} h)$.

Let $\tau$ denote a set of the commutative binary operations on M such that $M = ([0,1],*,e,\theta,\wedge,\vee)$ is a $*$ - cone for every $* \in \tau$.

Let $A_\tau = (\mathcal{F}_R,B^{(\tau)})$ be an algebraic system on $\mathcal{F}_R$, where $B^{(\tau)}$ is the set of $*$ - extended operations of B for every $* \in \tau$.

*Proposition 4.3.* Let $\circ,\square \in B$ and $*,\# \in \tau$. If $*$ and $\#$ are distributive on M, and $\circ$ and $\square$ distributive operations on $R^1$, then $\overset{(*)}{\circ}$ and $\overset{(\#)}{\square}$ are also distributive on $\mathcal{F}_R$.

*Proof.* On the one hand

$$[f \overset{(*)}{\circ} (g \overset{(\#)}{\square} h)](x) =$$

$$= f(x+a-[a\circ(b\Box c)])*(g\overset{(\#)}{\Box}h)(x+(b\Box c)-[a\circ(b\Box c)]) =$$

$$= f(x+a-[a\circ(b\Box c)])*\{g(x+b-[a\circ(b\Box c)])\#h(x+c-[a\circ(b\Box c)])\} =$$

$$= \{f(x+a-[(a\circ b)\Box(a\circ c)])*g(x+b-[(a\circ b)\Box(a\circ c)])\}\ \#$$

$$\#\ \{f(x+a-[(a\circ b)\Box(a\circ c)])*h(x+c-[(a\circ b)\Box(a\circ c)])\}.$$

On the other hand

$$[(f \overset{(*)}{\circ} g) \overset{(\#)}{\Box} (f \overset{(*)}{\circ} h)](x) =$$

$$=(f\overset{(*)}{\circ}g)(x+(a\circ b)-[(a\circ b)\Box(a\circ c)])\ \#$$

$$\#\ (f\overset{(*)}{\circ}h)(x+(a\circ c)-[(a\circ b)\Box(a\circ c)])$$

$$= \{f(x+a-[(a\circ b)\Box(a\circ c)])*g(x+b-[(a\circ b)\Box(a\circ c)])\}\ \#$$

$$\#\ \{f(x+a-[(a\circ b)\Box(a\circ c)])*h(x+c-[(a\circ b)\Box(a\circ c)])\}.$$ ■

*Corollary 4.4.*

$$f \overset{(*)}{\bullet} (g \overset{(\vee)}{+} h) = (f \overset{(*)}{\bullet} g) \overset{(\vee)}{+} (f \overset{(*)}{\bullet} h)$$

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