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RELATIONSHIP AMONG THREE TYPES OF CONVEX FUZZY SETS

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Abstract

This paper give a new characterization of convex fuzzy sets and relationship among types of convex fuzzy sets.

Key words: Convex fuasy set , strictly convex fuzzy set, 'strongly convex fuzzy set

1. INTRODUCTION

In the basic and classical paper [1], zaden developed a basic framework to treat mathematically the fuzzy phenomena or systoms which, due to intrinsic indefiniteness, cannot themselves be characterized precisely, and first introduced the important concept of fuzzy set. He pays special attention to the investigation of the convex fuzzy sets, strongly convex

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fuzzy sets and strictly convex fuzzy sets which convex nearly the second half of the space of the paper. Some properties of them were also studied by Brown [2], Weiss [3], katsaras and Liu[4], Lowen [5], and Liu[6]

This paper gives a new characterisation of convex fuzzy sets and relationship among three types of convex fuzzy sets. These results are very useful in fuzzy mathematical theory.

2. PRELIMINARIES

Throughout this paper E will denote the 1-dimensional Euclidean space R. As usual I will be used to denote the open unit interval. Fuzzy sets and values in I will be denoted by lower case Greek letters and we shall make no difference between notations for a fuzzy set with a constant value and that value itself.

DEFINITION 1. The fuzzy set λ on E is said to be convex fuzzy set iff for all $x,y \in E$ and $a \in I$,

$$\lambda(ax+(1-a)y) \geq \min(\lambda(x), \lambda(y))$$

DEFINITION 2. A fuzzy set λ on E is strongly convex fuzzy set iff for all $x, y \in E$ and $a \in I$,

$$\lambda(ax+(1-a)y) > \min(\lambda(x), \lambda(y))$$

DEFINITION 3. A fuzzy set λ on E is strictly convex fuzzy set iff for all $x, y \in E$, $\lambda(x) \neq \lambda(y)$ and $a \in I$,

$$\lambda(ax+(1-a)y) > \min(\lambda(x), \lambda(y))$$

Remark 1. Since I is the open unit interval, all above definition denote $x \neq y$.

Remark 2. The definition of strictly convex fuzzy set is an improved version for paper [1]. When $\lambda(x)$ is continuous, definitions here and [1] are equivalent.

3. A NEW CHARACTERISATION OF CONVEX FUZZY SET

In the paragraph we shall give a new characterisation of convex fuzzy set which is very important and basic in the following we shall investigate the relationship among three types of convex fuzzy sets.

THEORM 1. A fuzzy set λ on E is convex iff there exist two interval $I^{\leqslant}$, $I^{\geqslant}$ such that $E = I^{\leqslant} \cup I^{\geqslant}$, $I^{\leqslant}$ be left of $I^{\geqslant}$. λ be nondecreasing on $I^{\leqslant}$, λ be nonincreasing on $I^{\geqslant}$.

Proof. Suppose that fuzzy set λ be convex on E, let $I^{\leqslant} = \{x \in E \mid$ there is y, such that $x < y$ and $\lambda(x) < \lambda(y)\}$, $I^{\geqslant} = E \setminus I^{\leqslant}$.

First of, all, we prove λ be nondecreasing on $I^{\leqslant}$, λ be nonincreasing on $I^{\geqslant}$. Otherwise, there is $x_0 \in I^{\leqslant}$

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$z < x_0$, but $\lambda(z) > \lambda(x_0)$. Since $I^{\leqslant}$ definition, there is $y \in E$, such that $x_0 < y$, and $\lambda(x_0) < \lambda(y)$, then $\lambda(x_0) < \min(\lambda(z), \lambda(y))$, contradicting λ be convex fuzzy set. If $x_0 \in I^{\geqslant}$, $z > x_0$, $\lambda(z) > \lambda(x_0)$, then $x_0 \in I^{\leqslant}$ contradicting $I^{\geqslant} = E \setminus I^{\leqslant}$.

Next prove $I^{\leqslant}$, $I^{\geqslant}$ are interval.

Suppose that $x_0 \in I^{\leqslant}$, as before, for any $z < x_0$, we have $\lambda(z) \leqslant \lambda(x_0)$. Since $I^{\leqslant}$ definition, there is $y \in E$, such that $x_0 < y$, $\lambda(x_0) < \lambda(y)$, thus $\lambda(z) < \lambda(y)$ that is $z \in I^{\leqslant}$, then $I^{\leqslant}$ be interval. Since $I^{\geqslant} = E \setminus I^{\leqslant}$, we obtain also $I^{\geqslant}$ be interval.

Converse be obvious.

4. MAIN RESULTS

In the section we will investigate relationship of convex fuzzy set, strongly convex fuzzy set and strictly convex fuzzy set.

Lemma 1. Suppose that λ be convex fuzzy set on E, if $x_1 < x_2 < x_3$, $\lambda(x_1) > \lambda(x_2) = \lambda(x_3)$ ($\lambda(x_1) < \lambda(x_2) = \lambda(x_3)$) or $\lambda(x_1) = \lambda(x_2) > \lambda(x_3)$ ($\lambda(x_1) = \lambda(x_2) < \lambda(x_3)$), then λ be constant on $[x_2, x_3]$ or $[x_1, x_2]$.

Proof. From the argument of theorem 1 it is not difficult to see that if $x_1 < x_2$, $\lambda(x_1) > \lambda(x_2)$, we have $x_2 \in I^{\leqslant}$. NOw by $I^{\geqslant}$ definition, for any $x \in [x_2, x_3]$, $\lambda(x_2) \geqslant \lambda(x) \geqslant \lambda(x_3)$, since $\lambda(x_2) = \lambda(x_3)$, thus for any $x \in [x_2, x_3]$, we have $\lambda(x) = \lambda(x_2) = \lambda(x_3) =$ constant, rest prove is analogous.

Lemma 2. Suppose that be convex fuzzy set on E, if $x_1 < x_2 < x_3$, $\lambda(x_1) = \lambda(x_2) = \lambda(x_3)$, then λ be constant on $[x_1, x_2]$ or $[x_2, x_3]$ or $[x_1, x_3]$.

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Proof. Analogous to the proof of Lemma 1.

THEOREM 2. Let λ be fuzzy strongly convex set on E, then λ must be fuzzy convex on E.

Proof. Obviously.

THEOREM 3. Suppose that λ be convex fuzzy set on E, and λ attains its maximum on E at no more than one point, then λ be fuzzy strongly convex set on E.

Proof. Suppose that λ is not fuzzy strongly convex set, then by definition 2 , exist x_1, x_0, x_2: $x_1 < x_0 < x_2$ such that $\lambda(x_0) \leq \min(\lambda(x_1), \lambda(x_2))$. Without loss of generality we can suppose that $\lambda(x_1) \geqslant \lambda(x_2)$, thus $\lambda(x_0) \leqslant \lambda(x_2)$, On the other hand, since λ be fuzzy convex set, we have $\lambda(x_0) \geqslant \lambda(x_2)$. so $\lambda(x_0) = \lambda(x_2)$. If $\lambda(x_1) = \lambda(x_2)$ by Lemma 2 , λ be constant on $[x_1, x_0]$ or $[x_0, x_2]$ or $[x_1, x_2]$, this contradicts that λ attains its maximum on E no more than one point. If $\lambda(x_1) > \lambda(x_2)$, since $\lambda(x_0) = \lambda(x_2)$, by Lemma 1, λ be constant on $[x_0, x_2]$. This contradicts that λ attains its maximum on E no more than one point.

THEOREM 4. Suppose that λ be convex fuzzy set on E, and every local maximum of λ is a global maximum of λ on E, then λ be strictly convex fuzzy set on E.

Proof. Suppose that λ is not fuzzy strictly convex set, then by Definition 3, there exist x_1, x_2, x_0: $x_1 < x_0 < x_2$, such that $\lambda(x_1) \neq \lambda(x_2)$ and $\lambda(x_0) \leqslant \min(\lambda(x_1), \lambda(x_2))$

Without loss of generality we can suppose that $\lambda(x_1) < \lambda(x_2)$, thus $\lambda(x_0) \leqslant \lambda(x_1)$. On the other hand since λ be fuzzy convex set, we have $\lambda(x_0) \geqslant \lambda(x_1)$. So $\lambda(x_0) = \lambda(x_1)$. By Lemma 1, λ be constant $\lambda(x_1)$ on

$[x_1, x_0]$, while $\lambda(x_1) < \lambda(x_2)$. This contradicts that every local maximum of λ is a global maximum of λ on E.

THEOREM 5. Suppose that λ be fuzzy strictly convex set on E, and λ be upper semicontinuous on E, then λ be fuzzy convex set on E.

Proof. Suppose that $x_1, x_2 \in E$, if $\lambda(x_1) \neq \lambda(x_2)$, then by Definition 3, for any $a \in I$ we have $\lambda(ax_1+(1-a)x_2) > \min(\lambda(x_1), \lambda(x_2))$. Now suppose that $\lambda(x_1) = \lambda(x_2)$, if λ be not fuzzy convex set, by Definition 1, there exist $a_0 \in I$, such that $\lambda(a_0x_1+(1-a_0)x_2) < \lambda(x_1)$, let $\bar{x} = a_0x_1+(1-a_0)x_2$, because λ be upper semicontinuous, thus there exist $b_0 \in I$ such that $\lambda(\bar{x}) < \lambda(b_0x_1+(1-b_0)\bar{x}) < \lambda(x_1) = \lambda(x_2)$......(I) Notice $\bar{x}$ be convex combination of $b_0x_1+(1-b_0)\bar{x}$ and x_2. Thus by Definition 3 and $\lambda(b_0x_1+(1-b_0)\bar{x}) < \lambda(x_2)$ we have $\lambda(\bar{x}) > \lambda(b_0x_1+(1-b_0)\bar{x})$. This contradicts that (I).

THEOREM 6. Let λ be fuzzy strongly convex set on E, then λ must be fuzzy strictly convex set on E

Proof. Obviously.

Remark. Above result be not true in [1]

THEOREM 7. Suppose that λ be fuzzy strictly convex set on E, and λ attains its maximum on E at no more than one point, then λ be fuzzy strongly convex set.

Proof. Suppose that $x_1, x_2 \in E$, $x_1 \neq x_2$, $a \in I$

(a) If $\lambda(x_1) \neq \lambda(x_2)$, then by Definition 3

$$\lambda(ax_1+(1-a)x_2) > \min(\lambda(x_1), \lambda(x_2))$$

(b) If $\lambda(x_1) = \lambda(x_2)$, without loss of generality we can suppose that $x_1 < x_2$.

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(i) If there exist x_0: $x_1 < x_0 < x_2$, such that

$$\lambda(x_0) > \lambda(x_1) = \lambda(x_2),$$

then by λ be fuzzy strictly convex set, we obtain

$$\lambda(ax_1+(1-a)x_0) > \lambda(x_1) \text{ and } \lambda(ax_0+(1-a)x_2) > \lambda(x_2)$$

thus $\lambda(ax_1+(1-a)x_2) > \min(\lambda(x_1), \lambda(x_2))$

(ii) If $\lambda(ax_1+(1-a)x_2) \leq \lambda(x_1) = \lambda(x_2)$ ......(II),
then there exist $x_3 < x_1$ or $x_2 < x_3$ such that
$\lambda(x_3) > \lambda(x_1) = \lambda(x_2)$, (Otherwise, x_1, x_2 are all maximum point of λ, this contradicts that suppose of Theorem.) since λ be fuzzy strictly convex set, we have $\lambda(ax_3+(1-a)x_2) > \lambda(x_2)$ $(x_3 < x_1 < x_2)$ or

$$\lambda(ax_1+(1-a)x_3) > \lambda(x_1) \quad (x_1 < x_2 < x_3)$$

thus, we always have $\lambda(ax_1+(1-a)x_2) > \min(\lambda(x_1), \lambda(x_2))$
This contradicts (II)

REFERENCES

1. L.A.ZADEH, Fuzzy sets, inform, and control 8 (1965),338-353
2. J.G.Brown, A note on fuzzy sets, inform. and control 18(1971),32-39
3. M.D.Weiss, Fixed points, separation and induced topologies for fuzzy sets, J. Math. Anal. Appl. 50 142-150
4. A.K.Katsaras and D.B. Liu, Fuzzy vector spaces and fuzzy topological vector spaces, J. Math. Anal. Appl. 58(1977),135-146.
5. R.Lowen, convex fuzzy sets, Fuzzy sets and systems 3(1980), 291-310.
6. Y.-M.Liu, Some properties of convex fuzzy sets, J.Math Anal. Appl. 111(1985),119-129.
7. M.Prevot, Fuzzy goals under fyzzy constraints, J Math [illegible]