A METHOD TO CONSTRUCT t-NORMS*

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Abstract In this paper we give a method to construct t-norms from t-norms. Our procedure is motivated by the Appendix of [4].

1. Introduction

It is well-known from the literature that t-norms and t-conorms are used very often in fuzzy set theory (see e.g. [5] for references). Applications to practical problems require to use, in a sense, the 'most appropriate' t-norm or t-conorm. For this reason, the construction of new t-norms seems to be an important tool not only for the theory but also for the applications.

In the Appendix of [4] Schweizer and Sklar presented a method to construct t-norms from t-norms. In the present paper we generalize their procedure with the aid of some results of [5] and [2].

2. Background

Let $I = [0,1]$ and $I_0 = (0,1)$. In this paper we will use the terminology of [5] , i.e. ,

a) a function $T : I \times I \to I$ is said to be a *t-norm* iff T is commutative, associative, non-decreasing and

(1) $$T(a,1) = a \quad \forall\, a \in I.$$

A t-norm T is *Archimedean* iff T is continuous and

(2) $$T(a,a) < a \quad \forall\, a \in I_0.$$

An Archimedean t-norm T is *strict* iff T is strictly increasing in $I_0 \times I_0$.

* This work has been partially supported by OTKA-27-5-606.

b) a function $S : I \times I \to I$ is said to be a *t-conorm* or *Archimedean* or *strict* iff S has the same properties as T in part a) with the following modifications:

(1') $S(0,a) = a \quad \forall\, a \in I,$

(2') $S(a,a) > a \quad \forall\, a \in I_0.$

Assume now that T is a continuous t-norm and S is a continuous t-conorm. Then we can define the following operations in I (see [5]):

$$a\ \alpha_T\ b = \sup \{ x \ ; \ T(a,x) \leqq b \}$$

and

$$a\ \omega_S\ b = \inf \{ x \ ; \ b \leqq S(a,x) \}.$$

We list some properties of α_T and ω_S which will be used in the following part of the paper (more details can be found in [2] and [3]).

Lemma 1.

(a) If $b_1 \leqq b_2$ then $a\ \alpha_T\ b_1 \leqq a\ \alpha_T\ b_2$ and $a\ \omega_S\ b_1 \leqq a\ \omega_S\ b_2$.

(b) $S(a,\ a\ \omega_S\ b) \geqq b$; equality holds iff $a \leqq b$.

(c) $T(a,\ a\ \alpha_T\ b) \leqq b$; equality holds iff $a \geqq b$.

(d) If T is strict then $a\ \alpha_T\ T(a,b) = b$.

(e) If S is strict then $a\ \omega_S\ S(a,b) = b$.

(f) $a\ \omega_S\ 1 = a$ iff $a < 1$.

Proof. See [2] and [3].

We will also use the following result of Climescu [1].

Theorem 1. [1] Let (A,F) and (B,G) be semigroups. If the sets A and B are disjoint and if H is the mapping defined on $(A \cup B) \times (A \cup B)$ by

$$H(x,y) = \begin{cases} F(x,y) & \text{if } x \in A,\ y \in A \\ x & \text{if } x \in A,\ y \in B \\ y & \text{if } x \in B,\ y \in A \\ G(x,y) & \text{if } x \in B,\ y \in B \end{cases},$$

then $(A \cup B, H)$ is a semigroup.

4. The construction

Assume that T_1 , T_2 are any t-norms, T is a strict t-norm and S is a strict t-conorm. Moreover, let $\lambda \in I_0$. We can define two functions in the following way:

$$U_1(a,b) = T[\lambda , T_1(\lambda \; \alpha_T \; a , \lambda \; \alpha_T \; b)] \quad \text{for} \quad a,b \in [0,\lambda)$$

and

$$U_2(a,b) = S[\lambda , T_2(\lambda \; \omega_S \; a , \lambda \; \omega_S \; b)] \quad \text{for} \quad a,b \in [\lambda,1].$$

One can easily see that

(3) $$0 \leqq U_1(a,b) < \lambda \quad \text{for} \quad a,b \in [0,\lambda)$$

and

(4) $$\lambda \leqq U_2(a,b) \leqq 1 \quad \text{for} \quad a,b \in [\lambda,1].$$

Finally, let

(5) $$\tau(a,b) = \begin{cases} U_1(a,b) & \text{if } a \in [0,\lambda),\ b \in [0,\lambda) \\ \min(a,b) & \text{if } a \in [0,\lambda),\ b \in [\lambda,1] \\ \min(a,b) & \text{if } a \in [\lambda,1],\ b \in [0,\lambda) \\ U_2(a,b) & \text{if } a \in [\lambda,1],\ b \in [\lambda,1] \end{cases} .$$

Theorem 2.

(a) τ is a t-norm.

(b) If T_1 and T_2 are Archimedean then τ is Archimedean.

(c) If T_1 and T_2 are strict then τ is strict.

Proof.(a) Commutativity and monotonicity follow from (3) - (5).
$\tau(a,1) = a$ is obvious when $a \in [0,\lambda)$.
Assume now that $a \in [\lambda,1]$. Then
$\tau(a,1) = U_2(a,1) = S[\lambda , T_2(\lambda \; \omega_S \; a , \lambda \; \omega_S \; 1)] =$
$= S[\lambda , T_2(\lambda \; \omega_S \; a ,1)] = S(\lambda,\lambda \; \omega_S \; a) = a$ from Lemma 1 (b),(f).
Associativity: it is sufficient to show that U_1 and U_2 are associative (by Theorem 1 and (3),(4)). Using Lemma 1 (d) we get

$$\lambda \; \alpha_T \; T[\lambda , T_1(\lambda \; \alpha_T \; a , \lambda \; \alpha_T \; b)] = T_1(\lambda \; \alpha_T \; a , \lambda \; \alpha_T \; b).$$

So (by associativity of T_1) we have

$$U_1(U_1(a,b),c) = T[\lambda,\ T_1(\ \lambda\ \alpha_T\ T(\lambda,T_1(\lambda\ \alpha_T\ a\ ,\lambda\ \alpha_T\ b)))] =$$
$$= T[\lambda,\ T_1(T_1(\lambda\ \alpha_T\ a,\lambda\ \alpha_T\ b)\ ,\ \lambda\ \alpha_T\ c)] =$$
$$= T[\lambda,\ T_1(\lambda\ \alpha_T\ a\ ,\ T_1(\lambda\ \alpha_T\ b,\lambda\ \alpha_T\ c))] = U_1(a,U_1(b,c)).$$

It follows from Lemma 1 (e) that

$$\lambda\ \omega_S\ S[\lambda\ ,\ T_2(\lambda\ \omega_S\ a,\lambda\ \omega_S\ b)] = T_2(\lambda\ \omega_S\ a,\lambda\ \omega_S\ b).$$

Hence, one can see the associativity property for U_2 as for U_1.

Thus we have that τ is a t-norm.

(b) The reader can readily verify that τ is continuous when T_1 and T_2 are continuous. Moreover, if $a \in [0,\lambda)$ then

$$\tau(a,a) = U_1(a,a) = T[\lambda,\ T_1(\lambda\ \alpha_T\ a,\lambda\ \alpha_T\ a)] < T(\lambda,\lambda\ \alpha_T\ a) \leqq a$$

because T_1 is Archimedean.

If $a \in [\lambda,1]$ then

$$\tau(a,a) = U_2(a,a) = S[\lambda,\ T_2(\lambda\ \omega_S\ a,\lambda\ \omega_S\ a)] < S(\lambda,\lambda\ \omega_S\ a) = a$$

by Lemma 1 (b). Hence we get part (b).

(c) This part of the assertion follows from simple calculations. Thus our theorem is proved.

Example. Let $T_0(a,b) = ab$, $S_0(a,b) = a + b - ab$.

Then $$a\ \alpha_0\ b = \begin{cases} 1 & \text{if } a \leqq b \\ b/a & \text{if } a > b \end{cases}$$

and

$$a\ \omega_0\ b = \begin{cases} 0 & \text{if } a \geqq b \\ (b-a)/(1-a) & \text{if } a < b \end{cases}.$$

It is obvious that T and S are strict. One can conclude that Theorem 8 of [4] is a particular case of our Theorem 2 with t-norm T_0 and t-conorm S_0.

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