ON AN INTEGRAL OF FUZZY FUNCTIONS

Anna Paruleková Eva Rybáriková-Drobná
TECHNICAL UNIVERSITY
Vrbická 1944,031 01 Liptovský Mikuláš
CZECHOSLOVAKIA

In recent years the mathematic literature has been bringing approaches to fuzzy integral construction. The Polish mathematician Marian Matłoka in paper [1] generalized the theory of Riemann integral for fuzzy mapping, i.e. mapping with values in the set of fuzzy numbers. The paper deals with generalizing of Kurzweil integral for such a fuzzy mapping.

The book [2] describes Kurzweil theory of integral in great detail. Here we only introduce briefly the definition of integral in the Kurzweil sense.

Let $f:[a,b]\rightarrow R$ be a real function and $D=\{(E_i,x_i),i=1,\ldots n\}$ be such a partition of $[a,b]$ that $x_i\in E_i$, $E_i(i=1,\ldots,n)$ are compact subintervals of $[a,b]$ such that $\mathrm{Int}\ E_i\cap \mathrm{Int}\ E_j=\emptyset$ for $i\neq j(i,j=1,\ldots,n)$ and $\bigcup_{i=1}^{n} E_i=[a,b]$. The integral sum of f for the partition D have the form

$$S(f,D)=\sum_{i=1}^{n} f(x_i)\lambda(E_i) \qquad (1)$$

where λ is the Lebesgue measure.

Let $\Delta:[a,b]\longrightarrow(0,\infty)$ be a function. A convenient partition of $[a,b]$ with respect to Δ is such a partition D that

$$E_i\subset (x_i-\Delta(x_i),x_i+\Delta(x_i)),i=1,\ldots,n \qquad (2)$$

The set of all convenient partitions of $[a,b]$ with respect to Δ will be denoted by $\mathcal{D}(\Delta)$. It is easy to prove that for any function $\Delta : [a,b] \to (0,\infty)$ is $\mathcal{D}(\Delta) \neq \emptyset$.

Further we will consider a fuzzy mapping $f : [a,b] \to L(R)$ where $L(R)$ denote the set of fuzzy numbers, i.e. the set of functions $\mu : R \to [0,1]$ that satisfy the following conditions:

1) There is $x_0 \in R$ such that $\mu(x_0)=1$
2) The set $\mu_\alpha = \{x \in R : \mu(x) \geq \alpha\}$ is convex for all $\alpha \in (0,1]$
3) μ is upper semicontinuous function
4) There is a compact set $K \subset R$ such that $\{x \in R : \mu(x) > 0\} \subset K$.

Definition 1. Let $\mu \in L(R)$, $\nu \in L(R)$ and $\mu_\alpha = [a_\alpha, b_\alpha]$, $\nu_\alpha = [c_\alpha, d_\alpha]$ for $\forall \alpha \in (0,1]$.

i/ $\mu \leq \nu$, if $a_\alpha \leq c_\alpha$ and $b_\alpha \leq d_\alpha$ for all $\alpha \in (0,1]$

ii/ $\mu + \nu$ is the fuzzy number for which

$$(\mu + \nu)_\alpha = [a_\alpha + c_\alpha, b_\alpha + d_\alpha]$$

iii/ if $k \in R$, there $k\mu$ is the fuzzy number for which

$$(k\mu)_\alpha = [k a_\alpha, k b_\alpha], \quad k \geq 0$$

$$(k\mu)_\alpha = [k b_\alpha, k a_\alpha], \quad k < 0 \text{ for all } \alpha \in (0,1].$$

We can define on the set $L(R)$ a metric d by the following formula: $d(\mu, \nu) = \sup_{\alpha \in [0,1]} d(\mu_\alpha, \nu_\alpha)$, where $d(\mu_\alpha, \nu_\alpha) = d([a_\alpha, b_\alpha], [c_\alpha, d_\alpha]) = \max\{|c_\alpha - a_\alpha|, |d_\alpha - b_\alpha|\}$.

Puri and Ralescu proved that $(L(R), d)$ is a complete metric space.

Definition 2. A fuzzy mapping $f : [a,b] \to L(R)$ is integrable (in the Kurzweil sense), if

$$\exists c \in L(R)\ \forall \varepsilon > 0\ \exists \Delta : [a,b] \to (0,\infty) : \forall D \in \mathcal{D}(\Delta) : d(S(f,D), c) < \varepsilon.$$

The fuzzy number c is called the Kurzweil fuzzy integral of f and it is denoted by $\int_a^b f \, d\lambda$.

It is easy to prove that the following properties are satisfied: 1) $d(\alpha+\beta,\gamma+\delta) \leq d(\alpha,\gamma) + d(\beta,\delta)$

2) $d(\alpha,\beta) \leq d(\alpha+\gamma,\beta+\gamma)$

3) $d(k\alpha,k\beta) = k\,d(\alpha,\beta)$

4) $k(\alpha+\beta) = k\alpha + k\beta$

5) $0.\alpha = 0$

for all $\alpha,\beta,\gamma,\delta \in L(R)$ and all $k \in R$, $k>0$

6) if $k_1,\dots,k_n \in R$, $k_i>0$ $(i=1,\dots,n)$ are such that $\sum_{i=1}^{n} k_i = b-a$ and $\alpha_i, \beta_i \in L(R)$ such that $d(\alpha_i,\beta_i) < \varepsilon$ $(i=1,\dots,n)$, then $d(\sum_{i=1}^{n} k_i\alpha_i, \sum_{i=1}^{n} k_i\beta_i) < \varepsilon(b-a)$.

By the help of these properties the following properties of the Kurzweil fuzzy integral can be proved.

Proposition 3. If f,g are Kurzweil integrable fuzzy mappings and $\alpha,\beta \in R$, then $\alpha f+\beta g$ is Kurzweil integrable and $\int_a^b (\alpha f + \beta g)d\lambda = \alpha\int_a^b f d\lambda + \beta\int_a^b g d\lambda$ holds.

Proposition 4. A fuzzy mapping f is integrable iff

$$\forall \varepsilon>0\ \exists \Delta:[a,b] \to (0,\infty): \forall D_1, D_2 \in \mathcal{D}(\Delta): d(S(f,D_1),S(f,D_2)) < \varepsilon .$$

Proposition 5. If a fuzzy mapping f is integrable on $[a,b]$ then f is integrable on every subinterval $[c,d] \subset [a,b]$, too, and $\int_a^b f d\lambda = \int_a^c f d\lambda + \int_c^b f d\lambda$.

Proposition 6. If $(f_n)_{n=1}^{\infty}$ is a sequence of integrable fuzzy mappings on $[a,b]$, uniformly converging to a fuzzy mapping f then f is integrable, too, and

$$\lim_{n\to\infty} \int_a^b f_n d\lambda = \int_a^b f d\lambda \quad \text{holds.}$$

Remark 7. The uniform convergence of a sequence of fuzzy mappings $(f_n)_{n=1}^{\infty}$ on $[a,b]$ means the following:

$\forall \varepsilon>0\ \exists N_\varepsilon: \forall n>N_\varepsilon: d(f_n(x),f(x))<\varepsilon$ for all $x\in[a,b]$ ([3]).

For the Kurzweil fuzzy integral there holds the following theorem.

Theorem 8. Let $(f_n)_{n=1}^{\infty}$ be a sequence of integrable fuzzy mappings such that $f_n \leqslant f_{n+1}$ $(n=1,2,\ldots)$ and $\lim_{n\to\infty} f_n(x)=f(x)$ for all $x\in[a,b]$. Let the sequence of fuzzy numbers $\left(\int_a^b f_n d\lambda\right)_{n=1}^{\infty}$ is convergent. Then f is integrable fuzzy mapping, toe, and

$$\int_a^b f d\lambda = \lim_{n\to\infty} \int_a^b f_n d\lambda \quad \text{holds.}$$

Remark 9. $f_n \leqslant f_{n+}$ means that:
for all $x\in[a,b]$ is $f_n(x) \leqslant f_{n+1}(x)$ with respect to relation „ $\leqslant$ " on L(R).

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[3] Matłoka,M.: Fuzzy mappings - Sequences and Series, Busefal (in print).