A GENERAL EXTENSION THEOREM FOR MEASURES ON FUZZY SETS

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1. INTRODUCTION

A usual mathematical model for the quantum statistical mechanics is the quantum logic theory, i.e. the theory of orthomodular lattices ([5],[6]).

A state m on an orthomodular σ- complete lattice $L(\vee,\wedge,\perp, 0,1)$ is a mapping $m : L\to[0,1]$ ($[0,1]$ is the unit interval in the real line) satisfying the following two conditions:

1. $m(1) = 1$
2. if $a_i \leq a_j^{\perp}$, $i \neq j$, then $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$.

B. Riečan and A. Dvurečenskij pointed out ([1],[2]) that the same algebraic structure has the Piasecki P - measure ([3],[4]) and they introduced a new model for the statistical quantum mechanics. The Piasecki measure $m : M\to[0,1]$ is defined on an appropriate set of real functions $M\subset[0,1]^X$ and satisfies the following two conditions:

1. $m(f\vee f^{\perp}) = 1$ for every $f\in M$,
2. if $f_i \leq f_j^{\perp}$, $i \neq j$, then $m(\bigvee_{i=1}^{\infty} f_i) = \sum_{i=1}^{\infty} m(f_i)$.

Of course, here $f^{\perp} = 1 - f$.

In the present paper we work with the mapping $m : A\to[0,1]$ where A belongs to a class of lattices and in this way we obtain a common generalization of a state on a quantum logic as well as the Piasecki measure. The main result is a measure extension theorem. Special cases of our extension theorem is theorem on extension of states on logics [5] and the theorem on extension of the Piasecki measure [4].

2. THE MEASURE EXTENSION THEOREM

Let H be a σ- continuous lattice with 0 and 1 (i.e. if $x_n \nearrow x$, $y_n \nearrow y$, $n = 1,2,\ldots$, then $x_n \wedge y_n \nearrow x \wedge y$ and if $x_n \searrow x$, $y_n \searrow y$, $n = 1,2,\ldots$, then $x_n \vee y_n \searrow x \vee y$. Let $\perp : H \to H$ be a mapping satisfying the following conditions:

2.1 $(x^\perp)^\perp = x$ for every $x \in H$

2.2 if $x \leqq y$, then $y^\perp \leqq x^\perp$.

Let $A \subset H$ be such that

2.3 if $a,b \in A$, then $a \vee b \in A$

2.4 if $a \in A$, then $a^\perp \in A$

2.5 for every $x \in H$ there is a sequence $(a_n)_{n=1}^{\infty}$, $a_n \in A$, $n = 1,2,\ldots$, such that $x \leqq \bigvee_{n=1}^{\infty} a_n$.

There is given a mapping $m : A \to [0,1]$ satisfying the following conditions:

2.6 $m(a \vee a^\perp) = 1$ for every $a \in A$

2.7 if $a,b \in A$, $a \leqq b^\perp$, $a \neq b$, then $m(a \vee b) = m(a) + m(b)$

2.8 if $a_n \searrow a$, $b_n \nearrow b$, $a_n, b_n \in A$, $n = 1,2,\ldots$, $a \leqq b$, $a,b \in H$, then $m(a_n \wedge b_n^\perp) \searrow 0$ (m is a strongly continuous mapping)

2.9 if $a,b \in A$, then $m(a \vee b) + m(a \wedge b) = m(a) + m(b)$ (m is a valuation).

It is easy to prove that m has the following properties:

2.10 $m(a) + m(a^\perp) = 1$ for every $a \in A$

2.11 if $a,b \in A$, $a \leq b$, then $m(b) = m(a) + m(b \wedge a^{\perp})$

2.12 m is non - decreasing (i.e. if $a,b \in A$, $a \leq b$, then $m(a) \leq m(b)$

2.13 if $a_n \nearrow a$, $a_n, a \in A$, then $m(a_n) \nearrow m(a)$

2.14 if $b_n \searrow b$, $b_n, b \in A$, then $m(b_n) \searrow m(b)$

2.15 m is σ- additive (i.e. if $a_n \in A$, $n = 1,2,\dots$, $a_i \leq a_j^{\perp}$, $i \neq j$, $\bigvee_{n=1}^{\infty} a_n \in A$, then $m(\bigvee_{n=1}^{\infty} a_n) = \sum_{n=1}^{\infty} m(a_n)$)

2.16 m is subadditive (i.e. $m(a \vee b) \leq m(a) + m(b)$.

We want to extend the map m on the smallest σ- complete lattice S(A), generated by the sublattice A. Our extension will be made in the standard way.

Let $A^+ = \{ b \in H, \exists\, b_n \nearrow b,\ b_n \in A \}$ and $m^+ : A^+ \to [0,1]$,

$$m^+(b) = \lim_{n \to \infty} m(b_n) .$$

The definition of m^+ is correct, due to the following assertion.

Lemma. Let $a_n, b_n \in A$, $n = 1,2,\dots$, $a_n \nearrow a$, $b_n \nearrow b$, $a \leq b$, then

$$\lim_{n \to \infty} m(a_n) \leq \lim_{n \to \infty} m(b_n) .$$

It is not dificult to prove that m^+ is an extension of m, a valuation, it is upper continuous, non - decreasing, additive and subadditive map.

Let us define $m^*: H \to [0,1]$ in the following way:

$$m^*(x) = \inf \{ m^+(b),\ b \in A^+,\ b \geq x \}.$$

The main result is contained in the following theorem.

THEOREM. Let H be a σ- continuous lattice with 0 and 1, which satisfies the conditions 2.1 and 2.2. Let A be a sublattice of the lattice H which satisfies the conditions 2.3, 2.4, 2.5. Let $m : A \to [0,1]$ be the map satisfying the conditions 2.6 to 2.9. Let S(A) be the least σ- complete sublattice of the lattice H generated by the sublattice A.

Then there exists exactly one mapping $\overline{m} : S(A) \to [0,1]$ being an extension of m and satisfying 2.6 to 2.9. The mapping $\overline{m}$ coincides with m^* on $S(A)$.

REFERENCES

[1] Riečan, B.: A new approach to some notions of statistical quantum mechanics. Busefal.

[2] Dvurečenskij, A. - Riečan, B.: On randomness and fuzziness. In: Progress in fuzzy sets in Europe.

[3] Piasecki, K.: Probability of fuzzy events defined as denumerable additivity measure. Fuzzy sets and systems 17, 1985, 271 - 284.

[4] Piasecki, K.: Extension of fuzzy P - measure. Busefal 19, 1984, 26 - 41.

[5] Riečan, B.: The measure extension theorem for subadditive probability measures in orthomodular σ- continuous lattices. Comment. Math. Univ. Carol. 20, 2, 1979, 309 - 315.

[6] Varadarajan, V. S.: Geometry of quantum theory. Van Nostrand, New York 1968.

[7] Chovanec, F. - Kôpka, F.: Fasciculi Mathematici. To appear.