ON JOINT OBSERVABLE FOR F-QUANTUM SPACES

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1. F-quantum spaces

There are many analogies between fuzzy set theory and quantum logic models of quantum mechanics as it has been noticed in [7]. In a new axiomatic model for measurement of quantum mechanical observables based on fuzzy sets ideas we solve the problem of existence of a joint observable for a given system of observables of an F-quantum space.

Definition 1.1. By an F-quantum space we mean a couple (X,M), where X is a non-empty set and M is a subset of $[0,1]^X$ with (i) if $1(x) = 1$ for any $x \in X$, then $1 \in M$; (ii) if $f \in M$, then $1-f \in M$; (iii) if $1/2(x) = 1/2$ for any $x \in X$, then $1/2 \notin M$; (iv) if $\{f_n\} \subset M$, then $\bigvee_n f_n := \sup_n f_n \in M$. Using

Using the terminology of Piasecki [4] M is a soft fuzzy σ-algebra of fuzzy sets of X. Moreover, $\bigwedge_n f_n := \inf_n f_n \in M$ for any $\{f_n\} \subset M$. A non-empty subset $\mathcal{M} \subset M$ is said to be a Boolean algebra (σ-algebra) of an F-quantum space (X,M) if (i) there are the minimal and maximal elements $0_{\mathcal{M}}, 1_{\mathcal{M}} \in \mathcal{M}$ such that, for any $f \in \mathcal{M}$, $0_{\mathcal{M}} \le f \le 1_{\mathcal{M}}$; (ii) a mapping $\perp : f \longmapsto f^{\perp} = 1-f$, $f \in \mathcal{M}$, satisfies $f \vee f^{\perp} = 1_{\mathcal{M}}$ for any $f \in \mathcal{M}$; (iii) $\mathcal{M}$ is with respect to $\wedge, \vee, \perp$, $0_{\mathcal{M}}, 1_{\mathcal{M}}$ a Boolean algebra (σ-algebra). We note that $0_{\mathcal{M}} \neq 1_{\mathcal{M}}$, in the opposite case $0_{\mathcal{M}} = 1/2 \notin M$. It is simple that M is a Boolean σ-algebra iff $f = f^2$ for any $f \in M$, i.e, M consists from crisp sets.

Let $(\Omega, \mathcal{A})$ be a measurable space. We say that a mapping x:

$\mathcal{A} \to M$ is an $\mathcal{A}$-observable ($\mathcal{A}$-σ-observable) of (X,M) if (i) $x(A') = 1 - x(A)$ for any $A \in \mathcal{A}$ /here A' denotes the complement of A in Ω /; (ii) $x(\bigcup_i A_i) = \bigvee_i x(A_i)$ for any finite (countable) number of sets from $\mathcal{A}$. For the quantum mechanics it is of a great importnace the case when $\mathcal{A}$ is the Borel σ-algebra of some separable Banach space Y, in particular, when $Y = R_1$. In this case we call x an observable. The range of an (σ-) observable x, that is, the set $\mathcal{R}(x) = \{x(A) : A \in B(R_1)\}$ is a Boolean algebra (σ-algebra) of (X,M) with the minimal and maximal elements $x(\emptyset)$ and $x(R_1)$, respectively.

We say that a system $\{x_t : t \in T\}$, where x_t is a (σ-) observable of (X,M), has a joint (σ-) observable if there is an $R(T)$-observable ($B(T)$-σ-observable) x of (X,M), where $R(T)$ ($B(T)$) is the minimal algebra (σ-algebra) of R_1^T containing all finite-dimensional rectangles, such that $x(\bigcup_{t \in \alpha} A_t) = \bigvee_{t \in \alpha} x_t(A_t)$ for any $A_t \in B(R_1)$ and any finite non-empty $\alpha \subset T$.

2. Joint observables

Theorem 2.1. Let $\{x_t : t \in T\}$ be a system of (σ-) observables of an F-quantum space (X,M). The following assertions are equivalent: (i) $x_s(\emptyset) = x_t(\emptyset)$ for any $s,t \in T$.

(ii) $x_s(R_1) = x_t(R_1)$ for any $s,t \in T$.

(iii) Any subsystem $\{x_s, x_t\}$, $s,t \in T$ has a joint (σ-) obaservable.

(iv) $\{x_t : t \in T\}$ has a joint (σ-) observable.

Proof. The implication (iii) $\to$ (i) is evident. Conversely, for simplicity we put $x = x_s$, $y = x_t$. Denote by $\mathcal{D} = \{A \times B : A,B \in B(R_1)\}$ and define a mapping $h : \mathcal{D} \to M$ via $h(A \times B) = x(A) \wedge y(B)$, $A,B \in B(R_1)$. Using the standard techniques we may show that h is a $\mathcal{D}$-observable of (X,M). Analogicaly may be proved the implication (iii) $\to$ (iv).

For the case of σ-observables, the proof of (i) $\to$ (iv).

Since any Boolean algebra is isomorphic to some algebra of subsets [8], the conditions (i) -(ii) for observables are equivalent in order to exists a Boolean algebra $\mathcal{M} \subset M$ including all ranges $\mathcal{R}(x_t)$. Using the Zorn lemma, we may show that there is a maximal Boolean algebra $\mathcal{M}_o$ of (X,M) containing $\mathcal{M}$. We assert that $\mathcal{M}_o$ is a Boolean σ-algebra of (X,M). Indeed, let $\{f_n\} \subset \mathcal{M}_o$. Then $f = \bigvee_{n=1}^{\infty} f_n$ is an element of M, and $f \vee (1-f) = x_t(R_1)$ for any $t \in T$. In fact, $f \vee (1-f) = \max\{f, 1-f\} = (1 + |2f-1|)/2 = \lim_n (1+ |2g_n-1|)/2 = x_t(R_1)$, where $g_n = \bigvee_{i=1}^{n} f_i \in \mathcal{M}_o$, $n \geq 1$, and $g_n \vee (1-g_n) = x_t(R_1)$, $n \geq 1$. Define a Boolean algebra $\mathcal{M}_f = \{x_t(\emptyset), x_t(R_1), f, 1-f\}$ of (X,M). The maximality of $\mathcal{M}_o$ entails $\mathcal{M}_f \subset \mathcal{M}_o$.

The σ-distributivity of $\mathcal{M}_o$ entails that $\mathcal{M}_o$ has a σ-strong extension property, for definition see [8], According to Sikorski [8, Theorem 37.1], the joint observable x of X,M generated by $\{x_t : t \in T\}$ may be extended uniquely to a joint-σ-observable of (X,M). Q.E.D.

Let $f: R_1 \to R_1$ be a Borel measurable function and let x be a σ-observable of (X,M). Then by $f \circ x$ we mean a σ-observable of (X,M) defined by $f \circ x(E) = x(f^{-1}(E))$, $E \in B(R_1)$. According to the terminology of the theory of quantum logics (see [9]), we say that two σ-observables of (X,M) x and y are compatible if $x(\emptyset) = y(\emptyset)$.

Theorem 2.2. Let $\{x_t : t \in T\}$ be a system of σ-observables of an F-quantum space (X,M). The following assertions are equivalent: (i) $\{x_t : t \in T\}$ is a system of mutually compatible observables.

(ii) $\{x_t : t \in T\}$ has a joint σ-observable.

(iii) There is a measure space $(\Omega, \mathcal{S})$, an $\mathcal{S}$-measurable function $g_t : \Omega \to R_1$, and an $\mathcal{S}$-σ-observable h of (X,M) such that $h(g_t^{-1}(E)) = x_t(E)$ for all $t \in T$ and $E \in B(R_1)$.

If, moreover, (X,M) is separable in the sense that any Boolean σ-algebra of (X,M) has a countable generator, or that T is countable, then (i) is equivalent to (iv) There exists a σ-observable x and measurable functions $f_t: R_1 \longrightarrow R_1$ such that, for all $t \in T$, $x_t = f_t \circ x$.

Proof. It follows from Theorem 2.1 and Theorem 6.9 of [9]. Q.E.D.

Theorem 2.3. Let $x_1,\dots,x_n$ be mutually compatible σ-observables of an F quantum space (X,M) with a joint σ-observable x. If g is any real-valued Borel function on R_n, then $g\circ(x_1,\dots,x_n) : E \longmapsto x(g^{-1}(E))$, $E \in B(R_1)$, is a σ-observable of (X,M). If $g_1,\dots,g_k$ are real-valued Borel functions on R_n and $y_i = g_i\circ(x_1,\dots,x_n)$, then $y_1,\dots,y_k$ are mutually compatible σ-observables of (X,M), and for any real-valued Borel function h on R_k $h\circ(y_1,\dots,y_k) = (h(g_1,\dots,g_k))\circ(x_1,\dots,x_n)$, where $h(g_1,\dots,g_k)$ is the function $t = (t_1,\dots,t_n) \longmapsto h(g_1(t),\dots, g_k(t))$.

Proof. It is straightforward and therefore is omitted. Q.E.D.

Theorems 2.2 and 2.3 are of a great importance for building so-called functional calculus for compatible σ-observables. Therefore, for compatible σ-observables x and y of (X,M) we may define $x+y$, $x.y$, etc., if we put, for example, $x+y = (f+g)\circ z$, where $x = f\circ z$, $y = g\circ z$, according to Theorem 2.2, etc.

In the rest of this section we concentrate on the problem of existence of a joint distribution of a given system of mutually compatible σ-observables. For the quantum logic approach to quantum mechanics it is of great importance; it is known [2] that there are cases when it fails. By an F-state on F-quantum space (X,M) we understand a mapping $m: M \longrightarrow [0,1]$ such that $m(f\vee(1-f)) = 1$ for any $f \in M$; (ii) $m(\bigvee_i f_i) = \sum_i m(f_i)$ whenever $f_i \le 1-f_j$ for $i \ne j$. In the terminology of Piasecki [4] and F-state is a P-measure.

Theorem 2.4. Let $\{x_t: t\in T\}$ be a system of σ-observables of an F-quantum space (X,M). The pairwise compatibility of $\{x_t: t\in T\}$ implies that there exists a unique propability measure $\mu = \mu^m$ called on the product Borel σ-algebra $B(T)$ a joint distribution of $\{x_t: t\in T\}$ such that

$\mu(\prod_{t\in\alpha} A_t) = m(\bigwedge_{t\in\alpha} x_t(A_t))$ for any $A_t \in B(R_1)$, $t\in\alpha$, and any finite subset $\alpha \subset T$ in any F-state m.

Proof. According to Theorem 2.1, there is a joint σ-observable x of $\{x_t : t\in T\}$. Let us put $\mu(A) = m(x(A))$, $A\in B(T)$. We assert that $\mu = \mu^m$ is a probability measure in question. Q.E.D.

The authors hope to study also the problem of joint distribution for noncompatible observables.

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