CONDITIONAL EXPECTATION OF FUZZY RANDOM VARIABLE

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On the base of Aumann integral ([1]) of a set-valued function, M.L.Puri and D.A.Ralescu introduced in [5] the notion of a fuzzy random variable by the following way.
Let $F_o(R)$ denote the set of all fuzzy subsets $u:R \to \langle 0,1\rangle$ with the properties: (i) $u^\alpha = \{x \in R : u(x) \geq \alpha\}$ is compact for all $\alpha > 0$ and (ii) $u^1 = \{x \in R: u(x) = 1\} \neq \emptyset$.
Let (Ω, S, P) be a probability space where the probability measure P is assumed to be nonatomic.
Now, a fuzzy random variable is such a function $X: \Omega \to F_o(R)$ that $\{(\omega, x): x \in X^\alpha(\omega)\} \in S \times B(R)$ for every $\alpha \in \langle 0,1\rangle$ where $X^\alpha: \Omega \to 2^R$ is defined by $X^\alpha(\omega) = \{x \in R: X(\omega)(x) \geq \alpha\}$ and $B(R)$ is the Borel σ-algebra of R. A fuzzy random variable X is called integrably bounded if for every X^α there exists a function $h^\alpha: \Omega \to R$, $h^\alpha \in L^1(P)$ such that $|x| \leq h^\alpha(\omega)$ for all x, ω with $x \in X(\omega)$, $\alpha \in (0,1\rangle$. The family of all integrably bounded fuzzy variables we denote by $FV(\Omega)$.
Definition 1: For any fuzzy variable $X \in FV(\Omega)$ we define $\int_A X dP$, $A \in S$ as such $u \in F_o(R)$ for which $\{x \in R: u(x) \geq \alpha\} = (A)\int_A X^\alpha dP$, $\alpha \in (0,1\rangle$ where $(A)\int_A X^\alpha dP = \left\{ \int_A f dP,\ f \in L^1(P): f(\omega) \in X^\alpha(\omega)\right\}$ is Aumann integral

of X^{α}, $\alpha \in (0,1\rangle$, $A \in S$.

The proof of existence and uniqueness of this integral is quite the same as in [5] and is based on the following lemma.

Lemma 1: Let M be a set and let $\{M_{\alpha} : \alpha \in \langle 0,1\rangle\}$ be a family of subsets of M such that (i) $M_0 = M$, (ii) $\alpha \leq \beta$ implies $M_{\alpha} \supseteq M_{\beta}$ and (iii) $\alpha_1 \leq \alpha_2 \leq \dots \lim\limits_{n\to\infty} \alpha_n = \alpha$ implies $M_{\alpha} = \bigcap\limits_{n=1}^{\infty} M_{\alpha_n}$.

Then the function $\phi : M \to \langle 0,1\rangle$ defined by $\phi(x) = \sup\{\alpha \in \langle 0,1\rangle : x \in M_{\alpha}\}$ has the property that $\{x \in M : \phi(x) \geq \alpha\} = M_{\alpha}$ for every $\alpha \in \langle 0,1\rangle$.

Lemma 1 is proved in [4] and we shall use it to the construction of a conditional expectation of any integrably bounded fuzzy random variable.

Definition 2: Let S_0 be a sub-σ-algebra, $S_0 \subset S$ and $X : \Omega \to F_0(R)$ be an S-measurable (i.e. $\{(\omega,x) : x \in X^{\alpha}(\omega)\} \in S \times B(R), \alpha \in \langle 0,1\rangle$) integrably bounded fuzzy random variable. A conditional expectation of X relative to S_0 (let us write $E(X/S_0)$) is such a function $Y : \Omega \to F_0(R)$ that (i) Y is S_0-measurable and

$$\text{(ii)} \quad \int_A Y dP = \int_A X dP \quad \text{for all } A \in S_0$$

Theorem 1: Let $X \in FV(\Omega)$ be S-measurable and S_0 be a sub-σ-algebra of S. Then there exists such a $Y \in FV(\Omega)$ that Y is S_0-measurable and $\int_A X dP = \int_A Y dP$ for every $A \in S_0$.

The point how to prove this theorem is following.

Let $Z_{\alpha}(A) = (A)\int_A X^{\alpha} dP$, $\alpha \in (0,1\rangle$, $A \in S_0$. Every Z_{α}, $\alpha \in (0,1\rangle$ is a set-valued P-continuous measure of bounded variation and then, according to [2], Theorem 4.3., every Z_{α} has a Radon-Nikodým derivative F_{α} i.e. S_0-measurable set-valued function such that $Z_{\alpha}(A) = (A)\int_A F_{\alpha} dP$, $A \in S_0$. The functions F_{α} we can choose so that there exists $E \subset \Omega$ with $P(E) = 0$ and for every $\omega \in \Omega \setminus E$ a family

$\{F_\alpha(\omega), \alpha \in \langle 0,1\rangle\}$ satisfies the assumptions of Lemma 1 if we define $F_0(\omega) = R$, $\omega \in \Omega$. Define the function

$$Y(\omega) = \begin{cases} u \in F_0(R) \text{ where } u(x) = \sup\{\alpha : x \in F_\alpha(\omega)\} & \text{if } \omega \in \Omega \setminus E \\ v \in F_0(R) \text{ where } v(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} & \text{if } \omega \in E. \end{cases}$$

Now, Y is a version of the conditional expectation of X relative to S_0.

Let d denote the metric in the complete metric space $(F_0(R), d)$ introduced in [5]. Then the following theorem is true:

Theorem 2: Let $\{X_n\}_{n=1}^{\infty} \subset FV(\Omega)$ and $X \in FV(\Omega)$ be such that for every $\alpha \in (0,1\rangle$ X^α and X_n^α, $n=1,2,\ldots$ have compact and convex values and $X_n(\omega) \xrightarrow{d} X(\omega)$ for almost every $\omega \in \Omega$. Let S_0 be a sub-σ-algebra of S. Let there exist $g_1 \in L^1(P)$ such that $\sup\limits_{x \in X_n^\alpha(\omega)} |x| \leqslant g_1(\omega)$, $n \geqslant 1, \alpha > 0$ and $g_2 \in L^1(P)$ such that $\sup\limits_{x \in X^\alpha(\omega)} |x| \leqslant g_2(\omega)$ for $\alpha > 0$. Then $E(X_n/S_0)(\omega) \xrightarrow{d} E(X/S_0)(\omega)$ a.e.

References:

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