A NEW APPROACH TO SOME NOTIONS OF STATISTICAL QUANTUM MECHANICS

Beloslav RIEČAN, Liptovský Mikuláš

A usual mathematical model of statistical quantum mechanics is the quantum logic theory. Here we suggest to work instead of a quantum logic L with a family M of fuzzy subsets of a given set. This approach is motivated by a similarity between the notion of a state on a quantum logic ([1]) and the notion of the Piasecki measure ([2]) on a fuzzy algebra.

Example 1. Let $(L, \vee, \wedge, \perp, 0, 1)$ be a quantum logic. A state $m: L \to \langle 0, 1\rangle$ is a mapping such that

1. $m(1) = m(a \vee a^{\perp}) = 1$.
2. If $a_i \leq a_j^{\perp}$ $(i \neq j)$, then $m(\bigvee a_i) = \sum m(a_i)$.

Example 2. Let M be a set of fuzzy subsets of a given set (closed under some operations). Then the Piasecki measure $m: M \to \langle 0, 1\rangle$ is a mapping such that

1. $m(f \vee (1 - f)) = 1$ for every $f \in M$.
2. If $f_i \leq 1 - f_j$ $(i \neq j)$, then $m(\bigvee f_i) = \sum m(f_i)$.

Definition 1. By an F-quantum space we mean a couple (X, M), where X is a non-empty set and M is a subset $M \subset F(X)$ satisfying the following conditions:

1.1. If $e(x) = 1$ for every $x \in X$, then $e \in M$.

1.2. If $f \in M$, then $1 - f \in M$.

1.3. If $f_n \in M$ $(n=1,2,\dots)$, then $\bigvee f_n \in M$.

1.4. If $f(x) = \frac{1}{2}$ for every $x \in X$, then $f \notin M$.

Definition 2. By an F-state on an F-quantum space (X, M) we understand a mapping $m: M \to \langle 0, 1\rangle$ satisfying the following conditions:

2.1. $m(f \vee (1 - f)) = 1$ for every $f \in M$.

2.2. If $f_i \in M (i=1,2,\dots)$ and $f_i \leq 1 - f_j$ $(i \neq j)$, then

$$m(\bigvee f_i) = \sum m(f_i).$$

Definition 3. By an F-observable defined on an F-quantum space (X,M) we mean any mapping $Z: B(R^1) \longrightarrow M$ ($B(R^1)$ denotes the family of all Borel subsets of R^1) satisfying the following conditions:

3.1. $Z(A') = 1 - Z(A)$ for every $A \in B(R^1)$.

3.2. If $A_n \in B(R^1)$ $(n=1,2,\dots)$ and $A_n \cap A_m = \emptyset$ $(n \neq m)$, then

$$Z(\bigcup A_n) = \bigvee Z(A_n).$$

Definition 4. If Z is an F-observable defined on an F-quantum space (X,M), then by mean value of Z we mean

$$E(Z) = \int_R x \, dm_Z(x)$$

if this integral exists, $m_Z: B(R^1) \rightarrow \langle 0, 1 \rangle$ being the probability measure defined by the equality $m_Z(A) = m(Z(A))$.

Theorem. If (X,M) is an F-quantum space, then the system $S = \{ A \subset X \, ; \, \chi_A \in M \}$ is a σ-algebra. If m is an F-state on M, then $\overline{m}: S \rightarrow \langle 0, 1 \rangle$ defined by $\overline{m}(A) = m(\chi_A)$ is a probability measure. If $Y: X \rightarrow R$ is a random variable (with respect to the probability space $(X, S, \overline{m})$), then $Z_Y: B(R^1) \longrightarrow M$ defined by $Z_Y(A) = \chi_{Y^{-1}(A)}$ is an F-observable. It is integrable if and only if Y is. In this case

$$E(Z_Y) = E(Y) = \int_X Y \, d\overline{m}$$

From a mathematical point of view the following two problems seems to be important:

Problem 1. Characterize the set of all F-states on the space M of all measurable functions.

Problem 2. Find conditions under which some operations with F-observables can be defined, e.g. conditions under which to every F-observables Z_1, Z_2 there is an F-homomorphism $h: B(R^2) \longrightarrow M$ (i.e. $h(\bigcup A_n) = \bigvee h(A_n)$ and $h(A') = 1 - h(A)$) such that $h(A \times B) = Z_1(A) \wedge Z_2(B)$.

References

[1] Varadarajan, V.S.: Geometry of quantum theory. Van Nostrand, New York 1968.

[2] Piasecki, K.: On some relation between fuzzy probability measure and fuzzy P-measure. Busefal 23 (1985), 73 - 77.