EXTENDED FUZZY PREFERENCES (II): FUZZIFIED EXTENSION

F.J.Montero
Dept. Estadística e I.O.
Fac. CC. Matemáticas
Universidad Complutense
Madrid-28040 (Spain)

Abstract: This paper deals with decision-making problems where a fuzzy preference relation with no unfuzzy nondominated alternatives has been defined. While in the first part randomized extension was considered in order to get a solution, in this second part an aggregative approach to fuzzified extension will be developed.

Keywords: Fuzzy preference Relation, Decision Making, Nondominated Alternatives, SSB Utility Theory.

1. Introduction

As pointed out in the first part of this article, when a fuzzy preference relation has no unfuzzy nondominated alternatives, then a solution can be reached by considering an extended preference relation defined over the set of probability distributions. Now we shall consider an alternative model to such a randomized extension: the fuzzified extension, which will be defined over the set of fuzzy subsets. But it will be shown that only negative results can be obtained by an aggregative approach.

2. Fuzzified extensions

Given the fuzzy decision problem (X,μ), in this section we shall study the problem of how a fuzzy extended decision problem of how a fuzzy extended decision problem must be defined. In particular, the general framework is formalized, and the solution proposed by Orlovsky is analyzed.

Let us consider $X = \{x_1,\ldots,x_n\}$, and let us denote $\hat{X}$ the set of fuzzy subsets in X:

$$\check{X} = \{p = (p_1,\ldots,p_n)^t \in R^n / o \leq p_i \leq 1 \quad \forall i\}$$

Therefore, a fuzzified extended problem $(\check{X},\check{\mu})$ is to be considered, $\check{\mu}$ being an appropiate fuzzy preference relation over the set of fuzzy subsets $\check{X}$, that is, a mapping $\check{\mu} : \check{X} \times \check{X} \to |0,1|$. The coordenate p_i will represent the degree of membership of x_i to a fuzzy subset p, and the value $\check{\mu}(p,q)$ will be understood as the degree to which the fuzzy subset p is preferred to q.

Definition 3.- Let us denote $d^i \in \check{X}$ the unitary crisp set of alternative x_i ($d^i_j = 0 \quad \forall j \neq i,\ d^i_i = 1$). Any fuzzy preference relation $\check{\mu}$ over $\check{X}$ is said to be a fuzzified extension of a fuzzy preference relation μ over X, if $\check{\mu}(d^i, d^j) = \mu_{ij} \quad \forall i,j$.

As pointed above, the problem of extending a given fuzzy preference relation on a set, to a fuzzy preference relation **defined onto fuzzy subsets of such a set, has been introduced** by Orlovsky |8|. His proposed fuzzified extension is based on Zadeh's extension principle |10|, and it can be introduced by considering that the following two conditions can be accepted by the decision-maker:

$$\text{F1)}\quad \tilde{\mu}(p,q) = \max_i \min \{p_i, \tilde{\mu}(d^i,q)\}$$

$$\text{F2)}\quad \tilde{\mu}(p,q) = \max_j \min \{q_j, \tilde{\mu}(p,d^j)\}$$

These conditions can be understood as a fuzzified version of conditions R1 and R2 of the randomized extension. If both conditions F1 and F2 are assumed for the fuzzified extension, it follows that

$$\begin{aligned}\tilde{\mu}(p,q) &= \max_i \min \{p_i, \tilde{\mu}(d^i,q)\} = \\ &= \max_i \min \{p_i, \max_j \min \{q_j, \tilde{\mu}(d^i,d^j)\}\} = \\ &= \max_{i,j} \min (p_i, q_j, \mu_{ij}) \quad \forall p,q \in \hat{X}\end{aligned}$$

We shall denote $\tilde{\mu}_0$ Orlovsky's proposal. It is clear that $\tilde{\mu}_0(d^i,d^j) = \mu_{ij}$, and some desirable properties are proved in |9|. Moreover, if we denote $\tilde{X}_0^{UND}$ the set of unfuzzy nondominated fuzzy subsets under Orlovsky's extension,

$$\tilde{X}_0^{UND} = \{p \in \tilde{X} / \tilde{\mu}_0(p,q) \geq \tilde{\mu}_0(q,p) \quad \forall q \in \tilde{X}\}$$

it is easy to prove that $x_k \in X^{UND}$ if and only if $d^k \in \tilde{X}_0^{UND}$: since $x_k \in X^{UND}$ means that $\mu_{kj} \geq \mu_{jk}$ $\forall j$, then

$$\begin{aligned}\tilde{\mu}_0(d^k,q) &= \max_{i,j} \min (d^k_i, q_j, \mu_{ij}) = \\ &= \max_j \min (q_j, \mu_{kj}) \geq \max_j \min (q_j, \mu_{jk}) = \\ &= \max_{i,j} \min (q_j, d^k_i, \mu_{ji}) = \\ &= \tilde{\mu}_0(q,d^k)\end{aligned}$$

and therefore $d^k \in \tilde{X}_0^{UND}$ (sufficient condition is immediate because $\tilde{\mu}_0$ is an extension of μ).

But this approach have no sense in decision making problems: though $\tilde{\tilde{X}}_0^{UND}$ is assured to be non-empty, rejecting to make decision always will be inside our choice set (i.e., the fuzzy set $0 \in \tilde{\tilde{X}}$ such that $\emptyset_i = 0 \quad \forall i$ verifies that $\emptyset \in \tilde{\tilde{X}}_0^{UND}$ for any given fuzzy preference relation μ): trivially, $\tilde{\tilde{\mu}}_0(0,q) = \tilde{\tilde{\mu}}_0(q,0) = 0 \quad \forall q \in \tilde{\tilde{X}}$.

One could expect that such a problem will be avoided by considering the class of fuzzy partitions |2| instead of the class of all fuzzy subsets. In this sense, the extended problem will be $(\bar{X},\bar{\mu})$, where $\bar{X}$ is the set of fuzzy partitions of X,

$$\bar{X} = \{p = (p_1,\ldots,p_n)^t \in \mathbb{R}^n / \sum_{i=1}^{n} p_i = 1 \ , \ p_i \geq 0 \quad \forall i\}$$

and $\bar{\mu}$ is a fuzzy binary relation over it ($\hat{X} = \bar{X}$ from a formal point of view). Any mapping $\bar{\mu} : \bar{X} \times \bar{X} \to |0,1|$ such that $\bar{\mu}(d^i,d^j) = \mu_{ij}$ will be said a "restricted" fuzzified extension of μ.

As above, we can define $\bar{\mu}_0(r,q) = \hat{\mu}_0(p_1q) \quad \forall p,q \in \bar{X}$, but it also has no sense in decision making problems. The set of unfuzzy nondominated fuzzy partitions

$$\bar{X}_0^{UND} = \{p \in \bar{X}/\bar{\mu}_0 \ (p,q) \geq \bar{\mu}_0 \ (q,p) \quad \forall p \in \bar{X}\}$$

is assured to be non-empty, but the fuzzy partition $u = (1/n,\ldots,1/n)$ is always inside our choice set (i.e., $u \in \bar{X}_0^{UND}$ for any given fuzzy preference relation μ): since $\mu_{jj} = 1 \quad \forall j$, for any given $q \in \bar{X}$

$$1/n \geq \max_{i,j} \min \ (1/n,\ q_j,\ \mu_{ij}) \geq$$

$$\geq \max_{j} \min \ (1/n,\ q_j,\ \mu_{jj}) = \max_{j} \min \ (1/n,\ q_j) = 1/n$$

and therefore $\bar{\mu}_0(u,q) = 1/n$. Analogously, $\bar{\mu}_0(1,u) = 1/n$ holds, and hence $\bar{\mu}_0(u,q) = \bar{\mu}_0(q,u) \quad \forall q \in \bar{X}$.

In the paper of Baldwing and Guild |1| some methods of comparision of fuzzy sets are reviewed, also with Zadeh's extension principle as subyacent. Moreover, they consider $\mu(x_i,x_j) = f(x_i,x_j)$, where a function f is used to represent difference between the utilities of x_i and x_j.

3. Aggregative approach to fuzzified extension

It is clear that a trivial generalization of Orlovsky's idea of extension can be obtained by considering other aggregative operators instead of minimum or maximum. From an aggregative point of view, given a fuzzy preference relation μ defined on X and two fuzzy subsets $p,q \in \check{X}$, the minimum degrees of verification of preferences between each pair of alternatives x_i in p and x_j in q $(\min(p_i,q_j,\mu_{ij})\ \forall i,j)$ are aggregated through the max-operator. In this way, the global preference of p over q is obtained through two aggregation operators. But minimum operation is aggregating non-homogeneous quantities (p_i and q_j are related to fuzzy subsets and μ_{ij} is related to a fuzzy preference relation). Hence, it seems natural to consider two different aggregation operations: one in order to aggregate values related to fuzzy subsets in X, and other in order to aggregate values related to fuzzy preference relation on X. The first one can be defined as a mapping $*$: $|0,1| \times |0,1| \rightarrow |0,1|$, and if it is supposed to verify the rationality conditions proposed by Fung and Fu |8|, then it must be either the minimum operation, the maximum operation or either a mixed type operation (see |6| for a more general concept of aggregation rule).

If each value $p_i * q_j$ is understood as the weight or relative importance of the preference μ_{ij} between x_i and x_j , then the aggregation scheme proposed in Montero |7| can be considered. Basically, a mapping $\circledast$, verifying some rationality conditions and being compatible with $*$, must be defined in such a way that the aggregated preference is a fuzzified extended preferences: $\tilde{\mu}(p,q) = \underset{i,j}{\circledast} \mu_{ij}$. Each aggregation $\mu_{ij} * \mu_{km}$ is made depending on the weights $p_i * q_j$ and $p_k * q_m$. A formal approach is given in |7|, where the weighthed mean is axiomatically justified. In our case, the weighted mean rule is given by

$$\tilde{\mu}_w(p,q) = \sum_{i,j} (p_i * q_j) \cdot \mu_{ij} / \sum_{i,j} p_i * q_j$$

where $\sum_{i,j} p_i * q_j \neq 0$ has been supposed.

Between the rules axiomatically justified by Fung and Fu in |5|, only that based on the minimum ($p_i * q_j = \min(p_i, q_j)$ $\forall p_i, q_j \in |0,1|$) makes weighted mean $\hat{\mu}_w$ be a fuzzified extension (i.e., $\check{\mu}_w(d^i, d^j) = \mu_{ij}$) . In other words, it ought to be

$$\check{\mu}_w(p,q) = \sum_{i,j} \min(p_i, q_j) \cdot \mu_{ij} / \sum_{i,j} \min(p_i, q_j)$$

when Fung-Fu's conditions for the aggregation rule * are assumed.

Such a weighted mean $\check{\mu}_w(p,q)$ is defined only of $p \neq 0$ and $q \neq 0$. But since the fuzzy set $0(x) = 0 \quad \forall x \in X$ is not a real choice, one could accept that fuzzied extended preferences must be defined as a fuzzy preference relation on $\check{X}-\{0\}$, not on the whole $\check{X}$. In any case, such a problem of indefinition is avoided in the context of restricted fuzzied extensions.

In this way, the associated set of unfuzzy nondominated fuzzy subsets is

$$\check{X}_w^{UND} = \{p \in \check{X}-\{0\} / \check{\mu}_w(p,q) \geq \check{\mu}_w(q,p) \quad \forall q \in \check{X}-\{0\}\}$$

and the associated set of unfuzzy non dominated fuzzy partition, is given by

$$\bar{X}_w^{UND} = \{p \in \bar{X} / \check{\mu}_w(p,q) \geq \check{\mu}_w(q,p) \quad \forall q \in \bar{X}\}$$

As expected, an alternative x_k belongs to X^{UND} if and only if $d^k \in \check{X}_w^{UND}$, and analogous property holds for $\bar{X}_w^{UND}$. Due to the minimum operation, properties of such a fuzzy extended preference are developed with difficulty. But it is easy to find examples showing that the desired results $\check{X}_w^{UND} \neq \emptyset$ and $\bar{X}_w^{UND} \neq \emptyset$ does not necessarily hold.

4. Concluding remarks

Both extensions, randomized and fuzzified, can be considered in order to solve any decision problem with no unfuzzy non-dominated alternative. But it must be pointed out that any result obtained must be understood only as a help to make a better decision (see Freeling |4| for a general introduction to fuzzy decision analysis). It still remains the problem of how to make exact choices on the basis of vague preferences (a recent approach to such a problem can be seen in Dutta et al. |3|).

On one hand, it has been clarified the deep link between randomized extension of a fuzzy preference relation and the SSB utility theory. On the other hand, two types of fuzzified extension have been considered: that based on Zadeh's extension principle, and that based on the idea of aggregation of preferences. While randomized extension assures the existence of unfuzzy nondominated none of the two fuzzified extensions considered here can be applied in a general decision making problem. Though restricted fuzzified extension seems easier to understood than the non-restricted one, another approach to the concept of fuzzified extension must be tried.

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