THE FUZZY NONSINGULAR MATRIX

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ABSTRACT

Definitions of the row rank, column rank and Schein rank of a fuzzy matrix are given in [1].

Many papers on fuzzy mathematics established the these problems (as [1] — [6]) and success fully given an algorithem for solving the row rank, the column rank and the Schein rank of any fuzzy matrix.

In this paper we put forward a concept of the fuzzy nonsingular matrix and discusse preliminaly its propertes.

Keywords: The fuzzy nonsingular matrix, The fuzzy matrix of a full row rank, The fuzzy matrix of a full column rank.

I FUNDAMENTAL CONCEPTS

Let fuzzy matrices $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$ and $k \in [0,1]$. The sum of the two fuzzy matrices, the scalar product of a number and a fuzzy matrix, and the relation "$\leq$" of fuzzy matrices are defined respectively as follows:

$$A + B = (a_{ij} + b_{ij})_{m \times n} = (\max\{a_{ij}, b_{ij}\})_{m \times n} ;$$

$$kA = (ka_{ij})_{m \times n} = (\min\{k, a_{ij}\})_{m \times n} ;$$

$$A \leq B \quad \text{iff} \quad \forall i,j, \quad a_{ij} \leq b_{ij}.$$

The product of two fuzzy matrices ($A = (a_{ij})_{m \times t}$ and B =

$(b_{ij})_{t \times n})$ is defined as follows:

$$AB = (c_{ij})_{m \times n} = (\sum_{k=1}^{t} a_{ik}b_{kj})_{m \times n}$$

Under the addition and scalar product the set of all n-ary fuzzy row (column) vectors forms a fuzzy semilinear space, denoted by $V_n(V^n)$.

A vector set $\{X_1,\ldots,X_t\} \subseteq V_n(V^n)$ is independent if and only if there is not $X_i \in \{X_1,\ldots,X_t\}$ $(i=1,\ldots,t)$ such that it is represented as a linear combination of elements of $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_t$. If there is some $X_i \in \{X_1,\ldots,X_t\}$ $(1 \leq i \leq t)$ such that it is a liniear combination of elements of $X_1,\ldots,X_{i-1},X_{i+1},\ldots,X_t$, then set $\{X_1,\ldots,X_t\}$ it said to be dependent.

The set of linear combination of all column vectors of a $m \times n$ fuzzy matrix A is a subspace of V^n, denoted by C(A).

The rank $\rho_c(A)$ of C(A) is number of vectors in minimum generating set of C(A), i.e. Dim C(A) = $\rho_c(A)$.

Analogously define the row space R(A) and the row rank $\rho_r(A)$ of A.

A fuzzy matrix A is said to be of rank r if $\rho_r(A) = \rho_c(A) = r$, writed by $\rho(A)$.

The Schein rank $\rho_s(A)$ of a fuzzy matrix A is the least number of rank 1 matrices shose sum is A.

II FUZZY NONSINGULAR MATRIX

Definition 2.1 A $m \times n$ fuzzy matrix A is said to be nonsingular if $\rho_r(A) = m$ and $\rho_c(A) = n$.

Definition 2.2 A fuzzy permutation matrix is a square fuzzy matrix such that every row and every column contain exactly one 1, and other elements are 0.

Definition 2.3 A fuzzy square matrix B is said to be an inverse of a fuzzy square matrix A if $AB = BA = I$, where I is the unit matrix.

By definition for a fuzzy matrix of order $n \times n$, we have that

Theorem 2.1 A fuzzy matrix A of order $n \times n$ is a nonsingular if and only if all rows of A are linear independent and all columns of A are also linear independent.

Theorem 2.2 A fuzzy matrix A of order $n \times n$ is a nonsingular if and only if $\rho_r(A) = \rho_c(A) = \rho(A) = n$.

Therefore By the theorem 3.2 in paper [7] we give that:

Theorem 2.3 A fuzzy martix A of order $n \times n$ is a nonsingular if and only if $\rho_s(A) = n$.

Proposition 2.1 Let A is a $n \times n$ fuzzy permutation matrix. And let B is an arbitrary $n \times n$ fuzzy matrix. Then $\rho_r(AB) = \rho_r(BA) = \rho_r(B)$, $\rho_c(AB) = \rho_c(BA) = \rho_c(B)$, $\rho_s(AB) = \rho_s(BA) = \rho_s(B)$. If a fuzzy matrix B has a rank, then $\rho(AB) = \rho(BA) = \rho(B)$.

Theorem 2.4 Let A is a pemutation matrix, then A' is an inveres of A.

Proof. Let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}, \quad A\,A' = D = \begin{bmatrix} d_{11} & \dots & d_{1n} \\ \dots & \dots & \dots \\ d_{n1} & \dots & d_{nn} \end{bmatrix}.$$

Since every row of A constains exactly one 1, so that

$$d_{ii} = \sum_{k=1}^{n} a_{ik}\, a_{ik} = 1 \ (i=1,\dots,n).$$

Let in row i of A the t-th element is 1. If $i \neq j$, since in conlumn j of A' the t-th element is not 1. So that

$$d_{ij} = \sum_{k=1}^{n} a_{ik} a_{jk} = 0 \ (i \neq j,\ i,j=1,\dots,n).$$

Therefore $D = I$. Similarly may prove that $A'\,A = I$. Therefore A' is an inverse of A.

<u>Theorem 2.5</u> A fuzzy matrix A there is an inverse if and only if A is permutation matrix.

Proof. $\Leftarrow$ By the theorem 2.4 we have that

$$A'A = AA' = I.$$

Therefore A there is an inverse.

$\Rightarrow$ If A has an inverse B, $AB = BA = I$. Suppose that A is not a permutation matrix.

(i) If some row of A not only contains one 1, but also contains one non-zero element. Without loss of generality, we let the row 1 of A is $(1,a,0,\dots,0)$, $(0 < a \leqslant 1)$. Since $AB = I$, thus the column 1 of B is $(1,0,*\dots*)'$, and the column 2, ... , the column n of B are $(0\ 0\ *\dots*)'$, i.e.

$$B = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ * & * & \dots & * \\ \dots & \dots & \dots & \dots \\ * & * & \dots & * \end{bmatrix}$$

So that

$$BA = \begin{bmatrix} * & \cdots & * \\ 0 & \cdots & 0 \\ * & \cdots & * \\ \cdots & \cdots & \cdots \\ * & \cdots & * \end{bmatrix} \neq I.$$

This is contradiction.

(ii) If some row of A not only contains one 1 , but also contains non-zero elements, similarly may porve that there exist a contradiction.

(iii) If some row of A contains not 1, then for any B, $AB \neq I$.

Therefore every row of A contains exactly one 1, and other elements are zero.

Analogously may prove every column of A contains exactly 1, and other elements are zero.

Therefore A is a permutation matrix.

By this theorem we see that a fuzzy matrix with there is an inverse, in fact, is a Boolean matrix.

In fuzzy mathematics "a nonsingular fuzzy square matrix" and "fuzzy matrix with has inverse matrix" are not equivalent concept .

For example

$$A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}$$

is a nonsingular matrix, but it there is not an inverse matrix.

On the contrary we have that

Theorem 2.6 If a fuzzy square matrix there is an inverse matrix, then it is a nonsingular matrix.

In fact, a fuzzy square matrix A there is an inverse if and only if A is a pemutation matrix. And every pemutation matrix is a full row rank and full column rank. Therefore A is a nonsingular matrix.

Theorem 2.7 Let A is a $n\times n$ fuzzy nonsingular matrix and B is a $n\times n$ fuzzy matrix. If $\min_{i,j}\{a_{ij}\}\geq\max_{i,j}\{b_{ij}\}$, then $\rho_r(AB) = \rho_r(BA) = \rho_r(B)$, $\rho_c(AB) = \rho_c(BA) = \rho_c(B)$, $\rho_s(AB) = \rho_s(BA) = \rho_s(B)$. If B has the rank, then $\rho(AB) = \rho(BA) = \rho(B)$.

Proof. Obvious.

Theorem 2.8 Let A is a $m\times n$ nonsingular fuzzy matrix, and let B is a $n\times t$ fuzzy matrix. Then $\rho_c(AB)\leq \rho_c(B)$.

Proof. Let $\rho_c(B) = c$, and let it is $B = (B_1 \ldots B_c \ldots B_t)$, Without loss of generality, we suppose $\{B_1,\ldots,B_c\}$ is a basis of C(A). Then $B_1,\ldots,B_t$ are the linear combinations of $B_1,\ldots,B_c$. Let $B_i = \alpha_{i1}B_1 + \ldots + \alpha_{ic}B_c \quad (i=1, \ldots, t)$

Thus
$$AB_i = A(\alpha_{i1}B_1 + \ldots + \alpha_{ic}B_c)$$
$$= \alpha_{i1}AB_1 + \ldots + \alpha_{ic}AB_c \quad (i=1,\ldots,t)$$

That is $AB_1, \ldots, AB_t$ are the linear combinations of $AB_1, \ldots, AB_c$.

Therefore $\rho_c(AB)\leq c = \rho_c(B)$.

Notice that it is uncertain whether $\rho_c(AB) = \rho_c(B)$ or $\rho_c(AB) < \rho_c(B)$.

Example 2.1 Let

$$A = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 0.8 \\ 0.8 & 0.7 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.2 \\ 0.3 & 0.3 \end{bmatrix} .$$

Obvious A is a nonsingular fuzzy matrix and $1 = \rho_c(AB) < 2 = \rho_c(B)$. If let

$$C = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.7 \end{bmatrix}, \quad D = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} .$$

Then

$$CD = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.3 \\ 0.3 & 0.4 \end{bmatrix} .$$

Obvious C is a fuzzy nonsingular matrix and $\rho_c(CD) = \rho_c(D) = 2$.

Corollary Let A is a $m \times n$ fuzzy nonsingular matrix, and let B is a $t \times m$ fuzzy matrix. Then $\rho_r(BA) \leq \rho_r(B)$.

Let a $m \times n$ fuzzy marix is

$$A = \begin{bmatrix} X_1 \\ \vdots \\ X_m \end{bmatrix} = (Y_1 \ \ldots \ Y_n). \qquad (2.1)$$

all rows of A are a linear independent if and only if there extist not $X_i \in \{X_1, \ldots, X_m\}$ $(i=1,\ldots,m)$ such that it is repersented as a linear combination of elements of $\{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_m\}$, if and only if equations

$$X_i = x_1X_1 + \ldots + x_{i-1}X_{i-1} + x_{i+1}X_{i+1} + \ldots + x_mX_m \qquad (i=1, \ldots, m)$$

there are not a solution. If and only if fuzzy relational equations

$$(x_1 \;\cdots\; x_{i-1} \; x_{i+1} \;\cdots\; x_m)\begin{pmatrix} X_1 \\ \vdots \\ X_{i-1} \\ X_{i+1} \\ \vdots \\ X_m \end{pmatrix} = X_i \quad (i=1, \ldots ,m) \qquad (2.2)$$

there are not a solution.

Similarly all columns of A is a linear independent if and only if fuzzy ralational equations

$$(Y_1 \;\cdots\; Y_{j-1} \; Y_{j+1} \;\cdots\; Y_n)\begin{pmatrix} y_1 \\ \vdots \\ y_{j-1} \\ y_{j+1} \\ \vdots \\ y_n \end{pmatrix} = Y_j \; (j=1,\ldots,n) \qquad (2.3)$$

there are not a solution.

In summary we have

Theorem 2.9 Let as A as (2.1). A is a fuzzy nonsingular matrix if and only if fuzzy ralational equations (2.2) there are not a solution, and fuzzy ralational equations (2.3) there are also not a solution.

Example 2.2 Let

$$A = \begin{pmatrix} 1 & 0.9 & 0.8 \\ 0.9 & 0.8 & 0.7 \\ 0.8 & 0.7 & 0.6 \end{pmatrix}$$

Since

$$(x_2 \; x_3)\begin{pmatrix} 0.9 & 0.8 & 0.7 \\ 0.8 & 0.7 & 0.6 \end{pmatrix} = (\,1 \quad 0.9 \quad 0.8),$$

$$(x_1 \; x_3)\begin{pmatrix} 1 & 0.9 & 0.8 \\ 0.8 & 0.7 & 0.6 \end{pmatrix} = (0.9 \quad 0.8 \quad 0.7)$$

$$(x_1\ x_2)\begin{bmatrix}1 & 0.9 & 0.8\\ 0.9 & 0.8 & 0.7\end{bmatrix} = (0.8\ \ 0.7\ \ 0.6)$$

there are not a solution. And since

$$\begin{bmatrix}1 & 0.9\\ 0.9 & 0.8\\ 0.8 & 0.7\end{bmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{bmatrix}0.8\\ 0.7\\ 0.6\end{bmatrix} \quad ;$$

$$\begin{bmatrix}1 & 0.8\\ 0.9 & 0.7\\ 0.8 & 0.6\end{bmatrix}\begin{pmatrix}y_1\\ y_3\end{pmatrix} = \begin{bmatrix}0.9\\ 0.8\\ 0.7\end{bmatrix} \quad ;$$

$$\begin{bmatrix}0.9 & 0.8\\ 0.8 & 0.7\\ 0.7 & 0.6\end{bmatrix}\begin{pmatrix}y_2\\ y_3\end{pmatrix} = \begin{bmatrix}1\\ 0.9\\ 0.8\end{bmatrix}$$

there are not solution.

Therefore A is a nonsingular fuzzy matrix.

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