RELATION EQUATIONS IN RESIDUATED LATTICES

by

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1. INTRODUCTION

A generalization of classical boolean equation theory consists in lattice valued relation equation theory, [1] [2] [3][4]. In this note we shall be concerned with the problem of solving relation equations when the relations are valued on a lattice which is right-residuated under an isotone binary multiplication.
We extend in this framework the results stated by Luce. This Author in [2] solve the equation AX = B where A and B are boolean matrices and X is unknown. In [3] D.Rudeanu stressed that analogous results can be stated in a Brouwerian lattice. In this case also the greatest solution is found by E.Sanchez (see [4]).
Furthermore in this note we stated some results concerned with the set of solutions of the equation under discussion.

2. LATTICE RELATION EQUATIONS

Let L be a complete lattice which is defined on a binary multiplication satisfying the conditions: order-preserving i.e.

$a \leqslant b \Rightarrow xa \leqslant xb$ and $ax \leqslant bx$ for every $a,b,x \in L$; (1)

L is right-residuated under this multiplication, i.e. (2)
for $a,b \in L$ there exists the largest x such that $ax \leqslant b$; we shall denote such x by $a * b$.

If X is a non empty set, $F(X) = \{A: X \longrightarrow L\}$ is the set of L-sets (lattice valued sets) and if Y is another non empty set, we define L-relation every element of $F(X \times Y)$.

Let R(L) be the set of L-relations, in R(L) a partial order and two partial multiplications are defined like this:

for $A,B \in F(X \times Y)$ $A \leqslant B \Longleftrightarrow A(x,y) \leqslant B(x,y)$ for every $(x,y) \in X \times Y$; (3)

$$A\,B = C \Longleftrightarrow C(x,z) = \bigvee_{y \in Y} (A(x,y)\ B(y,z))$$

$$(A \circledast B) = D \Longleftrightarrow \bigwedge_{y \in Y} (A(x,y) * B(y,z)) \qquad (4)$$

where $A \in F(X \times Y)$, $B \in F(Y \times Z)$.

Lastly for $A \in F(X \times Y)$ the L-relation A^{-1}, inverse of A, is defined by $A^{-1} \in F(Y \times X)$ and $A^{-1}(y,x) = A(x,y)$. From (1) and (4) it follows:

Lemma 1. Let $A \in F(X \times Y)$ and $B \in F(X \times Y)$ $A \leqslant B$ then $AR \leqslant BR$ for every $R \in F(Y \times Z)$, and $R'A \leqslant R'B$ for every $R' \in F(Z \times X)$.

For $A \in F(X \times Y)$ and $B \in F(X \times Z)$, let us consider the

L-relation inequation

$$A\ H \leqslant B, \qquad (5)$$

and the L-relation equation

$$A\ H = B \qquad (6)$$

where H is unknown, let us denote by

$$I(A,B) = \{ R \in F(Y\times Z)\ /\ AX \leqslant B \}$$

and

$$S(A,B) = \{ R \in F(Y\times Z)\ /\ AX = B \}$$

the sets of solutions of (5) and (6) respectively.

Proposition 1. I(A,B) is non empty and there exists its largest element.

Proof. Let $M = (m_j)_{j\in I_m} = A^{-1} * B$, we will show M I(A,B).For $(x,z)\in Y\times Z$:

$$M(y,z) = \bigwedge_{x\in X} (A(x,y) * B(x,z)).$$

Hence from (1)

$$(AM)(x,z) = \bigvee_{y\in Y} (A(x,y)M(y,z)) \leqslant \bigvee_{y\in Y} (A(x,y).(A(x,y) * B(x,z)))$$

$$\leqslant \bigvee_{y\in Y} B(x,z) \quad \text{for every } (x,z)\in X\times Z.$$

Now if $R\in I(A,B)$, then $AR\leqslant B$, therefore

$$(A(x,y)\ R(y,z)) \leqslant B(x,z) \quad \text{for every } x\in X,\ y\in Y \text{ and } z\in Z,$$

from (2)

$$R(y,z) \leqslant A(x,y) * B(x,z), \quad \text{for every } x\in X,\ y\in Y,\ z\in Z,$$

and then

$$R(y,z) \leqslant \bigwedge_{x\in X} (A(x,y) * B(x,z)) = M(y,z).$$

Theorem 1. $S(A,B)$ is non empty iff $M = A^{-1} \circledast B = (\omega_j)_{j \in I_n} \in S(A,B)$.

Proof. Let $S(A,B) \neq \emptyset$ and $R \in S(A,B) \subseteq I(A,B)$, then $R \leqslant M$ and from

Lemma 1

$$B = AR \leq AM. \tag{7}$$

From (7) and Proposition 1 it follows $M \in S(A,B)$. The vice versa is obvious.

Corollary. If $S(A,B)$ is non empty then $A^{-1} \circledast B = \max S(A,B)$.

Proof. Trivial.

Proposition 2. Let L be a complete lattice with a binary multiplication which satisfies (1) and (2), then

$$a \bigvee_{w \in \Omega} x_w = \bigvee_{w \in \Omega} (ax_w). \tag{9}$$

Proof. Set $\bar{x} = \bigvee_{w \in \Omega} (ax_w)$, it is $ax_w \leqslant \bar{x}$; hence $x_w \leqslant a * \bar{x}$ for all $w \in \Omega$ and $\bigvee_{w \in \Omega} x_w \leqslant a * \bar{x}$. From (1) it follows:

$$a \bigvee_{w \in \Omega} x_w \leqslant a(a * \bar{x}) \leqslant \bar{x} = \bigvee_{w \in \Omega} (ax_w) \tag{10}$$

But $ax_w \leqslant a \bigvee_{w \in \Omega} x_w$ for all $w \in \Omega$ implies

$$\bigvee_{w \in \Omega} (ax_w) \leqslant a \bigvee_{w \in \Omega} x_w . \tag{11}$$

From (10) and (11) it follows (9).

Proposition 3. $S(A,B)$ is a join semilattice.

Proof. Indeed let $R, R' \in S(A,B)$ then for every $(x,z) \in X \times Z$

$$\bigvee_{y \in Y} (A(x,y)\ R(y,z)) = B(x,z) \tag{12}$$

also

$$\bigvee_{y \in Y} (A(x,y)\ R'\ (y,z)) = B(x,z). \qquad (13)$$

From (12) and (13)

$$\left(\bigvee_{y \in Y} (A(x,y)\ R(y,z))\right) \vee \left(\bigvee_{y \in Y} (A(x,y)\ R'(y,z))\right)$$

$$= \bigvee_{y \in Y} ((A(x,y))\ R(y,z)) \vee (A(x,y)\ R'\ (y,z))) = B(x,z),$$

by Proposition 2

$$\bigvee_{y \in Y} (A(x,y)\ (R(y,z) \vee R'(y,z))) = B(x,z)$$

which is equivalent to $R \vee R' \in S(A,B)$.

Proposition 4. If R', $R'' \in S(A,B)$ and $R' \leqslant R \leqslant R''$ then $R \in S(A,B)$.

Proof. Trivial from Lemma 1.

3. EQUATIONS OF FINITE RELATIONS

From now on we suppose that $X = \{x_1, \ldots, x_n\}$ $Y = \{y_1, \ldots, y_m\}$ $Z = \{z_1, \ldots, z_p\}$ where X,Y,Z are finite sets. We put $I_n = \{1, \ldots, n\}$ the set of first positive natural numbers. Let $A \in F(X \times Y)$, $R \in F(Y \times Z)$, $B \in F(X \times Z)$ be L-relations. For the sake of brevity we put: $A(x_i, y_j) = a_{ij}$, $R(y_j, z_k) = r_{jk}$, $B(x_i, z_k) = b_{ik}$ for every $i \in I_n$, $j \in I_m$, $k \in I_p$. Furthermore, for any $h \in I_p$ we denote by R_h the h-th column of R and by B_k the k-th column of B, for every $k \in I_p$. It is evident that R_h and B_k are elements of $F(Y \times \{z_h\})$ and $F(X \times \{z_k\})$, respectively. We observe that for any $k \in I_p$ we can consider the equation

$$A\ H = B_k \qquad (14)$$

where H is unknown. Then the problem of resolution of Eq. 6 leads back to solve p equations as (14). Therefore, we limit ourselves to the study of Eq. 6 when p=1. Thus every R F(Y×{z}) will be denoted by r_j for every $j \in I_m$ and every $B \in F(X \times \{z\})$ by b_i for every $i \in I_n$. Let $A = (a_{ij})$ $B = (b_i)$, $i \in I_n$, $j \in I_m$. For every matrix over L $\alpha = (\alpha_{ij})$ such that

$$\bigvee_{j \in I_m} \alpha_{ij} = b_i \qquad \forall\ i \in I_n \qquad (15)$$

we consider

$$H^{\alpha}_{ij} = \{x / a_{ij}x = \alpha_{ij}\} \text{ and}$$

$$H^{\alpha}_{j} = \bigcap_{i \in I_n} H^{\alpha}_{i} .$$

Theorem 2. S(A,B) is non empty iff there exists a matrix (α_{ij}) fulfilling (15) and such that $H^{\alpha}_{j} \neq \emptyset$ for every $j \in I_m$.

Proof. Let $S(A,B) \neq \emptyset$ and $R(r_1,\dots,r_m) \in S(A,B)$ then $\bigvee_{j \in I_m} a_{ij}\, r_j = b_i$ for every $i \in I_n$. Set $\alpha_{ij} = a_{ij} r_j$, the matrix (α_{ij}) fulfills (15); therefore $r_j \in H^{\alpha}_{j}$ that is not empty.

Conversely, let $\alpha = (\alpha_{ij})$ be a matrix fulfilling (15) with $H^{\alpha}_{j} \neq \emptyset$ for every $j \in I_m$, then any L-relation $R(r_1,\dots,r_m)$ with $r_j \in H^{\alpha}_{j}$ is obviously a solution of Eq. 6.

If we denote by $\Lambda(A,B)$ the set of the matrices $\alpha = (\alpha_{ij})$ fulfilling (15) with $H^{\alpha}_{j} \neq \emptyset$ for every $j \in I_m$, then the Theorem 2 characterizes the set of the solutions of Eq. 6 as

$$S(A,B) = \bigcup_{\alpha \in \Lambda(A,B)} (H^{\alpha}_{1} \times \dots \times H^{\alpha}_{n}).$$

We can verify that such a set, if not empty, has the largest element which is $A^{-1} \circledast B$. In fact, let $R = (r_1,\dots,r_m) \in \bigcup_{\alpha} (H_1^{\alpha} \times \dots \times H_m^{\alpha})$, then there exists a matrix α fulfilling (15) such that $a_{ij}\, r_j = \alpha_{ij} \leq b_i$ for every $i \in I_n$ and $j \in I_m$. It results $r_j \leq \bigwedge_{i \in I_n} (a_{ij} * b_i)$, that is $R \leq (A^{-1} \circledast B)$. We have still to prove that $(A^{-1} \circledast B)$ belongs to $\bigcup_{\alpha} (H_1^{\alpha} \times \dots \times H_m^{\alpha})$. For brevity's sake we set $\beta_j = \bigwedge_{i \in I_n} (a_{ij} * b_i)$ for every $j \in I_m$.

Let $\alpha_{ij} = a_{ij}\ \beta_j$ and $R \in S(A,B)$, we have

$$\bigvee_{j \in I_m} a_{ij}\, \beta_j \geq \bigvee_{j \in I_m} a_{ij}\, r_j = b_i \tag{16}$$

on the other hand

$$\bigvee_{j \in I_m} a_{ij}\, \beta_j \leq \bigvee_{j \in I_m} a_{ij} (a_{ij} * b_i) \leq \bigvee_{j \in I_m} b_i = b_i \tag{17}$$

from (16) and (17) it follows:

$$\bigvee_{j \in I_m} \alpha_{ij} = \bigvee_{j \in I_m} a_{ij}\, \beta_j = b_i$$

and, for this very definition of (α_{ij}), $\beta_j \in H_j^{\alpha}\ \forall j \in I_m$.

All that we have just proved is in accordance with the results of Theorem 1 and its Corollary.

The use of the Theorem 2 consists in bringing back the study of solutions of Eq. 6 to the one of the set $S^{\alpha}(A,B) = H_1^{\alpha} \times \dots \times H_m^{\alpha}$, for every $\alpha \in \Lambda$.

Let us observe that if $S(A,B) \neq \emptyset$ then for every $j \in I_m$ H_j^{α} has the greatest element $m_j^{\alpha} = \bigwedge_{i \in I_n} (a_{ij} * \alpha_{ij})$ and hence $m^{\alpha} = (m_j^{\alpha})_{j \in I_m}$ is the greatest element of $S^{\alpha}(A,B)$.

The following propositions hold:

Proposition 4. $S^{\alpha}(A,B)$ is a join-semilattice, for every $\alpha \in \Lambda$.

Proof. Trivial from Proposition 2.

Proposition 5. For every $\alpha \in \Lambda$, $S^{\alpha}(A,B)$ is a convex set .

Proof. Let $P = (p_j)_{j \in I_m}$, $Q = (q_j)_{j \in I_m}$, $P \leq R = (r_j)_{j \in I_m} \leq Q$ and $P, Q \in S^{\alpha}(A,B)$ then for every $i \in I_n$, $j \in I_m$ it is

$$a_{ij}p_j \leq a_{ij}r_j \leq a_{ij}q_j \Longrightarrow \alpha_{ij} \leq a_{ij}r_j \leq \alpha_{ij} \Longrightarrow$$
$$a_{ij}r_j = \alpha_{ij} \Longleftrightarrow R \in S^{\alpha}(A,B).$$

Let us put $\alpha^* = (\alpha^*_{ij})$ where $\alpha^* = a_{ij}m_j$ for every $i \in I_n$ and $j \in I_m$ we have the following

Corollary. Let $R' \in S(A,B)$, $R \in S^{\alpha^*}(A,B)$ and $R \leq R'$ then $R' \in S^{\alpha^*}(A,B)$.

Proposition 6. If for every $a,x,y \in L$ it is $a(x \wedge y) = ax \wedge ay$ then $S^{\alpha}(A,B)$ is a lattice, $\forall\ \alpha \in \Lambda$.

Proof. Let $P = (p_j)_{j \in I_m}$, $Q = (q_j)_{j \in I_m}$ and $P, Q \in S^{\alpha}(A,B)$ then for every $i \in I_n$ it is

$$\bigvee_j a_{ij}(p_j \wedge q_j) = \bigvee_j (a_{ij}p_j \wedge a_{ij}q_j) = \bigvee_j \alpha_{ij} = b_i,$$

so $(P \wedge Q) \in S^{\alpha}(A,B)$.

From that and from the Proposition 4 the thesis follows. Here we just stress that if L is a Brouwerian lattice, where xy is defined as $x \wedge y$ then for every $\alpha \in \Lambda$, $S^{\alpha}(A,B)$ is a lattice.

4. MINIMAL SOLUTIONS

Let us observe that if the set S(A,B) has minimal solutions, each of them is a minimal solution in the respective set $S^{\alpha}(A,B)$ to which it belongs. So, the minimal solutions of $S(A,B)$, if they exist, are to be looked for among the minimal elements of the sets $S^{\alpha}(A,B)$. The proposition below and its corollary give a partial answer to this problem.

Proposition 7. If $(\mu_j^{\alpha})_{j\in I_m}$ is a minimal element of $S^{\alpha}(A,B)$ and (α_{ij}) is a minimal element of $\Lambda(A,B)$, then $(\mu_j^{\alpha})_{j\in I_m}$ is a minimal element of $S(A,B)$.

Proof. Let $R = (r_1,\ldots,r_m) \in S(A,B)$ and $r_j \leq \mu_j^{\alpha}$ for every $j\in I_m$. If $R\in S^{\alpha'}(A,B)$ then from (1) it is $\alpha'_{ij} \leq \alpha_{ij}$ for every $i\in I_n$ and $j\in I_m$, by the hypotheses $\alpha'_{ij} = \alpha_{ij}$ for every $i\in I_n$ and $j\in I_m$. Then $R\in S^{\alpha}$ and $r_j = \mu_j^{\alpha}$ for any $j\in I_m$.

Corollary. For any matrix (α_{ij}) which is a minimal element of $\Lambda(A,B)$, it holds that $(\mu_j^{\alpha})_{j\in I_m}$ is a minimal element of $S^{\alpha}(A,B)$ if and only if it is a minimal element of $S(A,B)$.

Proof. Trivial.

One can wonder if the minimal solutions that Proposition 7 and its Corollary exhibit, exhaust the set of all minimal solutions of $S(A,B)$. Under the considered hypotheses we have not an answer to this question. However, it is possible to give an affirmative answer under the further condition of the finite distributivity of the product with respect to $\wedge$ operation, i.e. for every $a,x,y\in L$

$$a(x\wedge y) = ax \wedge ay.$$

Suppose that (18) is valid, then holds the

Proposition 8. If $(\mu_j^{\alpha})_{j \in I_m}$ is a minimal element of $S(A,B)$, then it is a minimal element of $S^{\alpha}(A,B)$ and $\alpha = (\alpha_{ij})$ is a minimal element of $\Lambda(A,B)$.

Proof. The first part of the thesis is obvious, let us prove the second one. Let $\alpha' = (\alpha'_{ij}) \in \Lambda(A,B)$ and $\alpha'_{ij} \leq \alpha_{ij}$ for every $i \in I_n$ and $j \in I_m$. Let us fix $j_o \in I_m$, then it results

$$b_i = \bigvee_{j \in I_m} \alpha'_{ij} \leq \alpha'_{ij_o} \vee \bigvee_{j \in I_m - \{j_o\}} \alpha_{ij} \leq \bigvee_{j \in I_m} \alpha_{ij} = b_i$$

for every $i \in I_n$, and

$$\alpha'_{ij_o} \vee \bigvee_{j \in I_m - \{j_o\}} \alpha_{ij} = b_i$$

for every $i \in I_n$. So the matrix $\alpha'' = (\alpha''_{ij})$ defined by

$$\alpha''_{ij} = \begin{cases} \alpha_{ij} & \text{if } j \in I_m - \{j_o\} \\ \alpha'_{ij} & \text{if } j = j_o \end{cases}$$

belongs to $\Lambda(A,B)$. $H_j^{\alpha''} = H_j^{\alpha}$ if $j \in I_m - \{j_o\}$ and $H_{j_o}^{\alpha''} = H_{j_o}^{\alpha'}$. Furthermore, $x \in H_{j_o}^{\alpha}$ and $y \in H_{j_o}^{\alpha'}$ imply

$$a_{ij_o}(x \wedge y) = a_{ij_o} x \wedge a_{ij_o} y = \alpha_{ij_o} \wedge \alpha'_{ij_o} = \alpha'_{ij_o} \quad i \in I_n,$$

so $\mu_{j_o}^{\alpha} \wedge r_{j_o} \in H_{j_o}^{\alpha'}$ for every $r_{j_o} \in H_{j_o}^{\alpha'}$. Then the relation $\bar{R} = (\bar{r}_1, \ldots, \bar{r}_m)$ defined as follows:

$$\bar{r}_j = \begin{cases} \mu_j^{\alpha} & \text{if } j \in I_m - \{j_o\} \\ \mu_j^{\alpha} \wedge r_{j_o} & \text{if } j = j_o \text{ and } r_{j_o} \in H_{j_o}^{\alpha'} \end{cases}$$

belongs to $S^{\alpha''}(A,B)$ and consequently it belongs to $S(A,B)$. But $\bar{r}_j \leq \mu_j^\alpha$ for every $j \in I_m$, thus from the minimality of $(\mu_j^\alpha)_{j \in I_m}$ in $S(A,B)$ we have

$\bar{r}_{j_o} = \mu_{j_o}^\alpha \wedge r_{j_o} = \mu_{j_o}^\alpha$ and then $\mu_{ij_o}^\alpha \leq r_{j_o}$ and

$\alpha_{ij_o} \leq \alpha'_{ij_o}$ i.e. $\alpha_{ij_o} = \alpha'_{ij_o}$ for every $i \in I_n$. From the

genericity of the index j_o the thesis follows.

The foregoing statements are utilized in the following:

Theorem 3. Let L be a complete lattice satisfying (1), (2) and (18), $R = (\mu_1^\alpha, \ldots, \mu_m^\alpha)$ $S^\alpha(A,B)$, then R is a minimal element of $S(A,B)$ if and only if it is a minimal element of $S^\alpha(A,B)$ and $\alpha = (\alpha_{ij})$ is a minimal element of $\Lambda(A,B)$.

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