MEASURES OF FUZZINESS OF FUZZY CLUSTERS

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Introduction

Fuzzy clustering algorithms yield fuzzy partitions as clustering solutions for partitioning finite data sets. One of the ways for evaluation of clustering results is to measure the amount of fuzziness of fuzzy partitions. Some measures of fuzziness of fuzzy clusters are given in [1] , [2] . The aim of this paper is to develope an axiomatic framework for the measures of fuzziness of fuzzy partitions and to demonstrate a simple mean for their constructing.

1. Preliminaries

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of vectors in p-dimensional Euclidian space; let us denote the class of all fuzzy sets built on X by L(X). L(X) can be partially ordered by relation $\leq_s$ called "sharpened" (De Luca and Termini [3]) defined for all $u, v \in L(X)$ as follows:

$$v \leq_s u \quad \text{iff} \quad v(x) \leq u(x) < \tfrac{1}{2} \quad \text{and} \quad v(x) \geq u(x) > \tfrac{1}{2} \qquad (1.1)$$

De Luca and Termini [3] introduced for every fuzzy set $u \in L(X)$ a measure d(u) of its "fuzziness". This measure satisfies the following conditions:

(1) $d(u) = 0$ iff for each $x \in X$: $u(x) = 0$ or $u(x) = 1$ (1.2a)

(2) $d(u)$ is maximum iff for each $x \in X$: $u(x) = \frac{1}{2}$ (1.2b)

(3) if $u, v \in L(X)$ are such that $v \leq_s u$ then $d(v) \leq d(u)$ (1.2c)

Let us denote the usual vector space of real $k \times m$ matrices by V_{km}. Partitions of X are defined in [2] as follows:

$$P_k = \Big\{ U \in V_{km};\ u_{ij} \in \{0,1\} \text{ for all } i,j;\ \sum_i u_{ij} = 1 \text{ for all } j;\ \sum_j u_{ij} > 0 \text{ for all } i \Big\}, \qquad (1.3)$$

$$P_{fk} = \Big\{ U \in V_{km};\ u_{ij} \in \langle 0,1 \rangle \text{ for all } i,j;\ \sum_i u_{ij} = 1 \text{ for all } j;\ \sum_j u_{ij} > 0 \text{ for all } i \Big\}. \qquad (1.4)$$

We call P_k the hard k-partition space and P_{fk} the fuzzy k-partition space associated with X. Bezdek [2] used so called classification entropy as a measure of fuzziness of $U \in P_{fk}$

$$H(U) = -\frac{1}{n} \sum_i \sum_j h(u_{ij}) , \qquad (1.5)$$

where $h(u_{ij}) = u_{ij} \cdot \log_a u_{ij}$ for $u_{ij} > 0$, $h(0) = 0$, a is a real constant, $a > 1$.

Backer [1] proposed many quantitative partitioning characterizations. Some of them are based upon the measures of fuzziness of fuzzy sets (by De Luca and Termini) , e.g. for $U \in P_{fk}$:

$$\varphi(U) = 1 - \frac{2}{k-1} \sum_{r=1}^{k-1} \sum_{s=r+1}^{k} d(u_r \cap u_s) \qquad (1.6)$$

letting $d(u_r \cap u_s)$ be a measure of fuzziness contained in intersection of fuzzy clusters u_r, u_s.

2. Sharpened fuzzy partitions

Definition 2.1.
Consider $U, W \in P_{fk}$. W is a sharpened version of U denoted by $W \prec U$ if and only if

$$w_{ij} \le u_{ij} < \frac{1}{k} \quad \text{and} \quad w_{ij} \ge u_{ij} > \frac{1}{k} . \qquad (2.1)$$

It is easy to show that relation $\prec$ satisfies the following properties:

i) P_{fk} is partially ordered by $\prec$

ii) $U = \left[\frac{1}{k}\right]$ is a minimal element of P_{fk}

iii) for each $U \in P_k$: if $V \prec U$ then $V = U$.

Theorem 2.1.
Partition $U \in P_{fk}$ has a sharpened version $V \in P_{fk} - \{U\}$ iff there exists $u_{ij} \in (0, \frac{1}{k} \rangle$.

Theorem 2.2.
Consider $U, W \in P_{fk}$, $W \prec U$. Then for $j=1,\dots,n$:

i) $$\sum_i (u_{ij} - \tfrac{1}{k})^2 \le \sum_i (w_{ij} - \tfrac{1}{k})^2 \qquad (2.2)$$

ii) $$\sum_{r=1}^{k-1} \sum_{s=r+1}^{k} |u_{rj} - u_{sj}| \quad \sum_{r=1}^{k-1} \sum_{s=r+1}^{k} |w_{rj} - w_{sj}| \qquad (2.3)$$

3. Measures of fuzziness of fuzzy partitions

Definition 3.1.
Consider a real nonnegative function $\varphi : P_{fk} \longrightarrow R$. This function is called a measure of fuzziness of partitions from P_{fk} iff the following properties hold true:

P1 $\varphi(U) = \varphi(U^{(p)})$, where $U^{(p)}$ is any permutation of U, (3.1a)

P2 $\varphi(U) = 0$ iff $U \in P_k$, (3.1b)

P3 $\varphi(U)$ is maximum iff $U = \left[\frac{1}{k}\right]$, (3.1c)

P4 if $W \prec U$ then $\varphi(W) \leq \varphi(U)$. (3.1d)

Theorem 3.1.
Let f be a real function on $\langle 0,1 \rangle$ satisfying one of two next conditions i), ii) and the condition iii):

i) $$A = \inf_{0 \leq x < y \leq \frac{1}{k}} \frac{f(y) - f(x)}{y - x} \geq \sup_{\frac{1}{k} \leq r < s \leq 1} \frac{f(s) - f(r)}{s - r} = B,$$

ii) $$C = \sup_{0 \leq x < y \leq \frac{1}{k}} \frac{f(y) - f(x)}{y - x} \leq \inf_{\frac{1}{k} \leq r < s \leq 1} \frac{f(s) - f(r)}{s - r} = D,$$

iii) $(A - B)^2 + (C - D)^2 \neq 0.$

Then there exist constants α, β so that
$$\varphi(U) = \alpha \sum_i \sum_j f(u_{ij}) + \beta \qquad (3.2)$$
is a measure of fuzziness of fuzzy partitions from P_{fk} and $\varphi(U) = 1$ for $U = \left[\frac{1}{k}\right]$.

Proof:
We prove the case $A \geq B$.
Let $V \prec U$. Then for j=1,...,n we have
$$\sum_i (f(u_{ij}) - f(v_{ij})) = \sum_{v_{ij} \geq \frac{1}{k}} (f(u_{ij}) - f(v_{ij})) + \sum_{v_{ij} < \frac{1}{k}} (f(u_{ij}) - f(v_{ij})) \geq$$
$$\geq A \cdot \sum_{v_{ij} \leq \frac{1}{k}} (u_{ij} - v_{ij}) + B \cdot \sum_{v_{ij} \geq \frac{1}{k}} (u_{ij} - v_{ij}) =$$

$$= B. \sum_i (u_{ij} - v_{ij}) + (A - B). \sum_{v_{ij} < \frac{1}{k}} (u_{ij} - v_{ij}) \geq 0,$$

so that $\sum_i \sum_j f(v_{ij}) \leq \sum_i \sum_j f(u_{ij})$.

Now it is enough to assure $\varphi(U) = 1$ for $U = \left[\frac{1}{k}\right]$ and $\varphi(W) = 0$ for $W \in P_k$, i.e.

$$\alpha.n.k.f\left(\frac{1}{k}\right) + \beta = 1$$

and
$$\alpha.n.\left(f(1) + (k-1).\, f(0)\right) + \beta = 0.$$

This system has a solution iff $k.f\left(\frac{1}{k}\right) \neq f(1) + (k-1).\, f(0)$.
If $A > B$, then we have:

$$k.f\left(\frac{1}{k}\right) - f(1) - (k-1).\, f(0) \geq (k-1).A.\,\frac{1}{k} - B.(1 - \frac{1}{k}) = \frac{k-1}{k}.(A-B) > 0.$$

If $A = B$, then $C \neq D$. From $C \geq A = B \geq D$ we get $C > A$ or $B > D$.
Let e.g. $C > A$. Then $f\left(\frac{1}{k}\right) - f(0) > A.\,\frac{1}{k}$, so that

$$k.f\left(\frac{1}{k}\right) - (k-1).\, f(0) - f(1) > (k-1).\, A.\,\frac{1}{k} - B.\left(1 - \frac{1}{k}\right) = 0.$$

Then the constants α , β are as follows:

$$\alpha = \frac{1}{n.\left[k.f\left(\frac{1}{k}\right) - (k-1).\, f(0) - f(1)\right]} ,$$

$$\beta = \left[- f(1) - (k-1).\, f(0)\right] .\alpha.n \quad .$$

Corollary 1.
Let $f: \langle 0,1\rangle \longrightarrow R$ be any nonconstant increasing on $\langle 0, \frac{1}{k}\rangle$ and decreasing on $\langle \frac{1}{k}, 1\rangle$ function. Then there exist constants α , β so that the function defined by equation (3.2) satisfies the properties P1 - P4 in Definition 3.1.

Corollary 2.
Let $f: \langle 0,1\rangle \longrightarrow R$ be any convex or any concave nonlinear function. Then there exist constants α , β so that the function defined by equation (3.2) satisfies the properties P1 - P4 in Definition 3.1.

Examples:
The following functions are measures of fuzziness of $U \in P_{fk}$:

a) $$\varphi_1(U) = 1 - \frac{k}{n(k-1)} \sum_i \sum_j \left(u_{ij} - \frac{1}{k}\right)^2 \qquad (3.3)$$

b) $$\varphi_2(U) = 1 - \frac{1}{n} \sum_i \sum_j u_{ij}^2 \qquad (3.4)$$

c) $\varphi_3(U) = H(U)$ (see (1.5)) (3.5)

d) $\varphi_4(U) = 1 - \frac{k}{2n(k-1)} \sum_i \sum_j \left| u_{ij} - \frac{1}{k} \right|$ (3.6)

e) $\varphi_5(U) = n - \sum_j \max_i u_{ij}$ (3.7)

The functions φ_1 - φ_4 are constructed by Theorem 3.1. Function φ_5 shows that there are also another means for constructing of measures of fuzziness of fuzzy partitions satisfying the properties P1 - P4 in Definition 3.1.

References:

[1] Backer, E.: Cluster analysis by optimal decomposition of induced fuzzy sets, Delftse Universitaire Press, Delft 1978

[2] Bezdek, J.C.: Pattern Recognition with Fuzzy Objective Function Algorithms, Plenum Press, New York 1981

[3] De Luca, A., Termini, S.: A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory, Inform. and Control 20 (1972) 301 - 312