FINITE ROUGH SETS AS PROBABILISTICLIKE FUZZY SETS.

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Pawlak [1] introduce the concept of <u>rough set</u> and he [2] show that the rough sets are not reducible to <u>fuzzy sets</u> [3] expressed by three-valued membership functions and the their basic set-theoretic operations are not compatible.

On the contrary, Wygralak [4] show that the rough set theory can be expressed in 3-valued fuzzy set theory language using <u>hetereogeneously</u> defined operations.

Here, finite rough sets are expressed by means of fuzzy sets and probabilisticlike operators.

Key-words. Triangular norms, probabilisticlike operators, fuzzy sets, rough sets, boolean, zadhean, pawlakean.

Notations. $\#A$ is the cardinality of the (crisp) set A;
$\bigvee$ is the <u>sup</u> (or <u>max</u>, if finitely) operator;
$\bigwedge$ is the <u>inf</u> (or <u>min</u>, if finitely) operator.

A. FUZZY SETS.

1. Fuzzy sets.

Let Ω be a universe of discourse, Λ a bounded lattice and $\Lambda^{\Omega} = \{\Omega \to \Lambda\}$ the totality of the maps from Ω to Λ .

<u>Def</u>. The 3-tuple $\tilde{F} = (\Omega, \Lambda, \Omega \xrightarrow{\mu} \Lambda)$ is called <u>zadhean fuzzy set</u> $\tilde{F}$ and the map μ is called <u>fuzzy membership function</u> of $\tilde{F}$.

The collection $\tilde{\mathcal{Z}}_{\Lambda}(\Omega) = \{(\Omega, \Lambda, \mu) | \mu \in \Lambda^{\Omega}\}$ is called <u>fuzzy zadhean</u> of Ω (i.e. the totality of the fuzzy sets $\tilde{F}$ in Ω in the sense of Λ).

<u>**Remarks**</u>. For short but improperly, we write also $\tilde{F} = (\Omega, \Lambda, \Omega \xrightarrow{\tilde{F}} \Lambda)$ where $\tilde{F}$ represent the fuzzy set and its membership function.

When we considere the subset $\Lambda_0 = \{0,1\} \subseteq \Lambda$, the 3-tuple $(\Omega, \Lambda, \Omega \to \Lambda_0)$ is called <u>cantorian fuzzy set</u> and the subset $\mathfrak{P}^{\circ}_{\Lambda}(\Omega) = \{(\Omega, \Lambda, \mu) \mid \mu \in \Lambda_0^{\Omega}\}$ is called <u>fuzzy boolean</u> of Ω.

Through isomorfisme, a cantorian fuzzy set is a <u>crisp</u> set of Ω and the fuzzy boolean is the <u>boolean</u> $\mathcal{B}(\Omega) = \{A \mid A \subseteq \Omega\}$ (totality of the parts of Ω).

Here, we considere the lattice $\Lambda = [0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$.

3. <u>Triangular norms</u> [5] .

<u>Def</u>. A binary operation $*$ in the real unit interval $[0,1]$, i.e. a function T: $[0,1] \times [0,1] \to [0,1]$, is called <u>triangular norm</u> (shortly, <u>t-norm</u>) iff it satisfay the following conditions:

i) $a * (b * c) = (a * b) * c$,

ii) $a * b = b * a$,

iii) $a * b \leq c * d$ if $a \leq c$ and $b \leq d$,

iv) $a * 0 = 0$ and $a * 1 = a$.

The dual binary operation $\circledast$, i.e. a function S : $[0,1] \times [0,1] \to [0,1]$, defined by: $a \circledast b = 1 - (1 - a) * (1 - b)$, is called <u>t-conorm</u> : it fulfills the conditions i)ii)iii) and iv') $a \circledast 1 = 1$, $a \circledast 0 = a$.

Moreover, if $*$ is distributive, then: $a \circledast b = a + b - a * b$.

<u>Examples</u>. Important examples of t-norms T and their t-conorms S are:

$$\text{E1)}\quad \begin{cases} T_w(a,b) = \begin{cases} a \wedge b & \text{if } a \vee b = 1 \\ 0 & \text{if } a \vee b < 1 \end{cases} \\ S_w(a,b) = \begin{cases} a \vee b & \text{if } a \wedge b = 0 \\ 1 & \text{if } a \wedge b > 0 \end{cases} \end{cases}$$

$$\text{E2)}\quad \begin{cases} T_z(a,b) = a \wedge b \\ S_z(a,b) = a \vee b \end{cases}$$

$$\text{E3)}\quad \begin{cases} T_b(a,b) = 0 \vee (a + b - 1) & \left[\underline{\text{bounded difference}} \quad \ominus\right] \\ S_b(a,b) = 1 \wedge (a + b) & \left[\underline{\text{bounded sum}} \quad \oplus\right] \end{cases}$$

$$\text{E4)}\quad \begin{cases} T_p(a,b) = a \cdot b & \left[\underline{\text{algebraic product}}\right] \\ S_p(a,b) = a + b - a.b & \left[\underline{\text{alg.(or probabilistic) sum}}\right] \end{cases}$$

<u>Remarks</u>. We can define a partial ordering $\prec$ on t-norms, with: $T \prec T'$ if $T(a,b) \leq T'(a,b)$ $\forall (a,b) \in [0,1] \times [0,1]$.

If $\mathcal{T}$ is the family of all t-norms, results:

$T_w \prec T \prec T_z \quad \forall T \in \mathcal{T}$;

and for the dual family $\mathcal{S}$:

$S_w \succ S \succ S_z \quad \forall S \in \mathcal{S}$.

In particular: $T_w \prec T_b \prec T_p \prec T_z \prec S_z \prec S_p \prec S_b \prec S_w$.

3. <u>Fuzzy set-theoretic operations</u>.

Let be $\tilde{\mathfrak{Z}} = \tilde{\mathfrak{Z}}_{\Lambda}(\Omega)$ the zadhean of Ω .

If $\tilde{A}, \tilde{B} \in \tilde{\mathfrak{Z}}$, we define:

$\tilde{A}'$ <u>complement</u> of $\tilde{A}$,

$\tilde{A} \cap \tilde{B}$ <u>intersection</u> of $\tilde{A}$ and $\tilde{B}$,

$\tilde{A} \cup \tilde{B}$ <u>union</u> of $\tilde{A}$ and $\tilde{B}$

in the following way:

$\tilde{A}' = \tilde{C}$ where $\tilde{C}(x) = 1 - \tilde{A}(x)$ $\forall\ x \in \Omega$,
$\tilde{A} \cap \tilde{B} = \tilde{D}$ where $\tilde{D}(x) = \tilde{A}(x) * \tilde{B}(x)$ $\forall\ x \in \Omega$,
$\tilde{A} \cup \tilde{B} = \tilde{E}$ where $E(x) = \tilde{A}(x) @ \tilde{B}(x)$ $\forall\ x \in \Omega$,
with $*$ and $@$ t-norm and dual t-conorm, respectively.

In particular, we have:

by	set-th. operator	expressed by	with algeb. operator
E2	$\cap$	t-norm T_z	$\wedge$
"	$\cup$	t-conorm S_z	$\vee$
E3	$\dot{\cap}$	t-norm T_b	$\ominus$
"	$\dot{\cup}$	t-conorm S_b	$\oplus$
E4	$\hat{\cdot}$	t-norm T_p	$\cdot$
"	$\hat{+}$	t-conorm S_p	$+$

Results: $\tilde{A} \dot{\cap} \tilde{B} \subseteq \tilde{A}\ \hat{\cdot}\ \tilde{B} \subseteq \tilde{A} \cap \tilde{B} \subseteq \tilde{A} \cup \tilde{B} \subseteq \tilde{A}\ \hat{+}\ \tilde{B} \subseteq \tilde{A} \dot{\cup} \tilde{B}$.
Zadeh in [3] use the t-norm T_z and t-conorm S_z.

4. Fuzzy [sub]spaces.

The algebraic system $\tilde{\mathfrak{F}} = (\tilde{\mathfrak{F}}, \cup, \cap, ')$ is called fuzzy space, while $(\tilde{\mathfrak{F}}^o, \cup, \cap, ')$ is called fuzzy subspace when $\tilde{\mathfrak{F}}^o \subseteq \tilde{\mathfrak{F}}$ and $\tilde{\mathfrak{F}}^o$ is closed for $\cup, \cap$ and $'$. The fuzzy [sub]space $(\tilde{\mathfrak{F}}, \hat{+}, \hat{\cdot}, ')$ is called probabilistic fuzzy [sub]space.

B. ROUGH SETS.

Let Ω be a universe of discourse, $\mathcal{B} = \mathcal{B}(\Omega)$ its boolean and $\Pi = \Pi(\Omega)$ the totality of its partitions.
It is know that if $\pi \in \Pi$ then is $\pi \subseteq \mathcal{B}$, i.e. $\exists$ the immersion map $i_\pi : \pi \rightarrow \mathcal{B}$ such that $i_\pi(\alpha) = \alpha$ $\forall\ \alpha \in \pi$.
Let $\pi \in \Pi$ be ($\mathcal{R}_\pi$ is the equivalence relation associated to π and, in this section, $\mathcal{R}_\pi$ is called indiscernibility relation).

Let $[\,]_{\pi}$ be the map $x \longrightarrow [x]_{\pi}$ so defined:
if $x \in \Omega\ \exists\ 1$, and only $1, \pi_x \in \pi$ such that $x \in \pi_x \subseteq \Omega$; $[x]_{\pi}$ is called <u>equivalence class</u> of x in the sense of π .

1.<u>Pawlak's rough sets</u>.

<u>Def</u>. If $\underline{P} \subseteq \overline{P}$ and $\underline{P},\overline{P} \subseteq \pi$, the couple $\ddot{P} = (\underline{P},\overline{P})$ is called Pawlak's <u>abstract rough set</u> (in Ω in the sense of π); while if $P \subseteq \Omega$, the subsets of Ω :

$$\underline{P} = \bigcup_{[x]_{\pi} \subseteq P} i_{\pi}([x]_{\pi}) \quad , \quad \overline{P} = \bigcup_{x \in P} i_{\pi}([x]_{\pi})$$

define the triple $\dddot{P} = (P, \underline{P}, \overline{P})$ called Pawlak's <u>concrete rough set</u> and the set P is called its <u>support</u>.

The set $\check{P} = \overline{P} - \underline{P}$ is called the <u>rough boundary</u>.

The couple $\Theta = (\Omega , \pi)$ is called <u>approximation space</u> of Ω in the sense of π .

The collections $\ddot{\wp}$ and $\dddot{\wp}$ of all $\ddot{P}$ and $\dddot{P}$ are called the <u>abstract</u> and <u>concrete pawlakean</u> of Ω, respectively.

2.<u>Rough set-theoretic operators</u>.

From here we only deal the concrete pawlakean $\dddot{\wp}$.

If $\dddot{A},\dddot{B} \in \dddot{\wp}$, results:

$\underline{A} \cup \underline{B} \subseteq \underline{A \cup B} \subseteq A \cup B \subseteq \overline{A} \cup \overline{B} = \overline{A \cup B}$ and

$\underline{A} \cap \underline{B} = \underline{A \cap B} \subseteq A \cap B \subseteq \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

From this inclusions it is clear that if we want to define in $\dddot{\wp}$ the set-theoretic operators "union", "intersection" and "complementation", we dont meet a natural formulation, because $\underline{A} \cup \underline{B} = \underline{A \cup B}$ and $\overline{A \cap B} = \overline{A} \cap \overline{B}$ not hold in general case. But if we privilege the support set and its set-theoretics operators, we can define in natural manner the rough operators $\sqcup$, $\sqcap$, $\neg$. We pose:

$$\neg \dddot{A} = \dddot{C} \quad \text{where } \dddot{C} = (\Omega - A, \Omega - \bar{A}, \Omega - \underline{A}),$$
$$\ddot{A} \sqcup \ddot{B} = \dddot{D} \quad \text{"} \quad \ddot{D} = (A \cup B, \underline{A \cup B}, \overline{A \cup B}),$$
$$\ddot{A} \sqcap \ddot{B} = \ddot{E} \quad \text{"} \quad \ddot{E} = (A \cap B, \underline{A \cap B}, \overline{A \cap B}).$$

The algebraic system $(\dddot{\mathcal{P}}, \sqcup, \sqcap, \neg)$ is called rough space.

C. ROUGH SETS AS FUZZY SETS.

In the rough and fuzzy set theory the main idea is the membership predicate. In this theories one can say that x surely or possibly belongs or surely not belongs: the difference between Pawlak's and Zadeh's theorys consiste in the various meanings of the word "possibly". For Pawlak the possible-membership values are ill-know, for Zadeh this values are know, gradual and notequals If we give a constant value for the rough-possible, the rough set-theoretic operators can modifier this value; on the contrary the max and min operators dont modifie the corresponding fuzzy-possible value. Consequently it is necessary to find a good link between rough-membership value and fuzzy-membership value.

1. Representation problem.

Let Ω be a universe of discourse and $\mathcal{B}$, $\dddot{\mathcal{P}}$, $\tilde{\mathcal{Z}}$ its boolean, pawlakean and zadehan, respectively. Moreover, $\dddot{\mathcal{E}} = (\dddot{\mathcal{P}}, \dddot{\cup}, \dddot{\cap}, \dddot{\neg})$ and $\tilde{\mathcal{F}} = (\tilde{\mathcal{Z}}, \tilde{\cup}, \tilde{\cap}, \tilde{\neg})$ are any associated algebraic systems.
If we want to find a connection between rough and fuzzy sets, we meet the classical representation

Problem: is it possible to find a map $\omega : \dddot{\mathcal{E}} \to \tilde{\mathcal{F}}$ such that:

- $\omega(\dddot{A}\, \dddot{\cup}\, \dddot{B}) = \omega(\dddot{A})\omega(\dddot{\cup})\omega(\dddot{B}) = \tilde{A}\, \tilde{\cup}\, \tilde{B}$
- $\omega(\dddot{A}\, \dddot{\cap}\, \dddot{B}) = \omega(\dddot{A})\, \omega(\dddot{\cap})\omega(\dddot{B}) = \tilde{A}\, \tilde{\cap}\, \tilde{B}$
- $\omega(\dddot{\neg}\, \dddot{A}) = \omega(\dddot{\neg})\omega(\dddot{A}) = \tilde{A}'$?

Naturally the starting point stand in the association between $\dddot{A}$ and $\tilde{A} = \omega(\ddot{A})$, whence it is necessary to find a good membership value in $\tilde{\mathcal{F}}$, when $x \in \underline{A}$, $x \in \check{A}$, $x \in \Omega - \bar{A}$.

Now we go to examine this problem through the Pawlak's and Wygralak's approachs.

A natural way consist in $\dddot{A} \xrightarrow{\omega} \tilde{A}$ where:

$$\tilde{A}(x) = \begin{cases} 1 & \text{when } x \in \underline{A} \\ 0 & \text{"} \quad x \notin \bar{A} \\ 1/2 & \text{"} \quad x \in \check{A}. \end{cases}$$

In this manner, Pawlak [2] examine the present problem in the case $(\ddot{\mathcal{P}}, \sqcup, \sqcap, \neg)$ and $(\tilde{\mathcal{F}}, \cup, \cap, ')$, where the answer is negative: he remarks that

$(\omega(A \sqcup B))(x) \neq \max(\tilde{A}(x), \tilde{B}(x))$ and
$(\omega(A \sqcap B))(x) \neq \min(\tilde{A}(x), \tilde{B}(x))$ in the general case.

Against this handicap, Wygralak [3] instead propose hetereogeneously defined operators.

2. Wygralak's rough operators.

Here we repeat the Wygralak's ~~notations~~ propositions and their notations, where U is the universe of discourse.

Proposition 1. - For any rough sets Y, Z and for every $x \in U$:

$$(Y \sqcap Z)(x) = \begin{cases} \max(0, Y(x)+Z(x)-1) & \text{if } Y(x)=Z(x)=1/2 \text{ and } [x]_R \cap (Y \cap Z) = \emptyset \\ \min(Y(x), Z(x)) & \text{otherwise.} \end{cases}$$

Proposition 2. - For any rough sets Y, Z and for every $x \in U$:

$$(Y \sqcup Z)(x) = \begin{cases} \min(1, Y(x)+Z(x)) & \text{if } Y(x)=Z(x)=1/2 \text{ and } [x]_R \subset Y \cup Z \\ \max(Y(x), Z(x)) & \text{otherwise.} \end{cases}$$

Because in 1Top and 2Top is al~~t~~ways $Y(x)+Z(x)=1$, we can give a semplified formulation of Wygralak's propositions.

$\forall\ \ddot{A},\ddot{B}\in\ddot{\mathcal{F}}$ and $\forall\ x\in\Omega$, if $\omega(\ddot{A}) = \tilde{A}$, results:

$$(\omega\ (\ddot{A}\sqcap\ddot{B}))(x) = \begin{cases} 0 & \text{if } x\notin\overline{A\cap B} \\ \tilde{A}(x)\wedge\tilde{B}(x) & \text{otherwise;} \end{cases}$$

$$(\omega\ (\ddot{A}\sqcup\ddot{B}))(x) = \begin{cases} 1 & \text{if } x\in\underline{A\cup B} \\ \tilde{A}(x)\vee\tilde{B}(x) & \text{otherwise.} \end{cases}$$

This formulation avoide the expression "heteregeneously defined operations".

3. Finite probabilisticlike fuzzy space.

Let Ω be a finite universe of discourse, $\mathcal{B}$ its boolean, $\tilde{\mathcal{F}}$ its zadehan and $\pi\in\Pi$ a partition of Ω .
$\forall\ F\in\mathcal{B}$ we pose this map: $\Omega\xrightarrow{\mu_F}\Lambda$ so defined:

$$\mu_F(x) = \frac{\#([x]_\pi\cap F)}{\#[x]_\pi}\qquad\forall x\in\Omega .$$

Results: $\tilde{F} = (\Omega\ ,\Lambda\ ,\mu_F)\in\tilde{\mathcal{F}}$ and $\tilde{F}' = (\Omega\ ,\Lambda\ ,\mu_{\Omega-F})$.
Moreover, if $A,B\in\mathcal{B}$ we can define a t-norm $\top$:

$$\mu_A(x)\top\mu_B(x) = \frac{\#([x]_\pi\cap\ (A\cap B))}{\#\ [x]_\pi}$$

and its t-conorm $\perp$:

$$\mu_A(x)\perp\mu_B(x) = \mu_A(x)+\mu_B(x) - (\mu_A(x)\top\mu_B(x)) =$$

$$= \frac{\#([x]_\pi\cap A)+\#([x]_\pi\cap B)-\#([x]_\pi\cap(A\cap B))}{\#[x]_\pi} .$$

The corresponding algebraic system $(\tilde{\mathcal{Z}}, \hat{\perp}, \hat{\top}, ')$ is a finite probabilisticlike fuzzy space, where in particular results: $\tilde{F}\hat{\perp}\tilde{F}' = \tilde{\Omega}$ and $\tilde{F}\hat{\top}\tilde{F}' = \tilde{\emptyset}$. In this space the fuzzy sets are expressed by 3-valued membership functions.

4. Rough sets as probabilisticlike fuzzy sets.

Let Ω be a finite universe of discourse and $\mathcal{B}, \dddot{\mathcal{P}}, \tilde{\mathcal{Z}}$ its boolean, pawlakean and zadehan, respectively; and moreover given the algebraic systems: $\dddot{\mathcal{E}} = (\dddot{\mathcal{P}}, \sqcup, \sqcap, \neg)$, $\tilde{\mathcal{F}} = (\tilde{\mathcal{Z}}, \hat{\perp}, \hat{\top}, ')$.

Proposition. There is a map $\omega : \dddot{\mathcal{E}} \longrightarrow \tilde{\mathcal{F}}$ such that:

1) $\omega(\dddot{A} \sqcup \dddot{B}) = \omega(\dddot{A})\,\omega(\sqcup)\,\omega(\dddot{B}) = \tilde{A}\hat{\perp}\tilde{B}$
2) $\omega(\dddot{A} \sqcap \dddot{B}) = \omega(\dddot{A})\,\omega(\sqcap)\,\omega(\dddot{B}) = \tilde{A}\hat{\top}\tilde{B}$
3) $\omega(\neg\dddot{A}) = \omega(\neg)\,\omega(\dddot{A}) = \tilde{A}'$.

If we write the diagram:

$$\begin{tikzcd} \mathcal{B} \arrow[r, "q"] \arrow[d, leftrightarrow, "j"'] & \tilde{\mathcal{Z}} \\ \dddot{\mathcal{P}} \arrow[ur, "\omega"'] & \end{tikzcd}$$

between $\mathcal{B}$ and $\dddot{\mathcal{P}}$ there is a 1-1 correspondance (trivially $\mathcal{B} \ni A \overset{j}{\longleftrightarrow} (A, \underline{A}, \bar{A}) \in \dddot{\mathcal{P}}$). Moreover, let q be the correspondance defined by $q(A) = \tilde{A}$, where its membership function is the map μ_A defined in 3 of this section:

$$x \xrightarrow{\mu_A} \frac{\#([x]_R \cap A)}{\#[x]_R} \qquad \forall x \in \Omega .$$

Results: $\omega(\dddot{A}) = q(j(\dddot{A})) = q(\dot{A}) = \tilde{A}$ and 1),2),3) are satisfied.

5. Conclusions.

For Pawlak and Wygralak, when x stand in the rough boundary, its value is allways constant and equal to 1/2; in this paper it is constant but only into the equivalence class. Practically if we have, into the indiscernibility class, for every element, a non-principal know information (a measure), we can try an approach to the discernibility.

D. REFERENCES.

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