STABILITY IN POSSIBILISTIC LINEAR EQUALITY SYSTEMS

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Linear equality systems with fuzzy parameters (and crisp variables) defined by the extension principle are called possibilistic linear equality systems. The study focuses on the problem of stability in these systems with respect to changes of fuzzy parameters.

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1. Introduction

Using mathematical models of reality we often encounter the problem of finding all solutions to the system of linear equations

$$a_{i1}x_1 + \ldots + a_{in}x_n = b_i \qquad i = \overline{1,m} \tag{1}$$

where a_{ij}, b_i are real numbers.
It is known that the problem (1) generally belongs to the class of ill-posed problems, so a small perturbation of parameters may cause a large deviation in the solution-set.
In this paper the system (1) with fuzzy numbers a_{ij}, b_i and crisp variables x_j is considered. It can be proved that the system (1) with Lipschitzian fuzzy numbers always belongs to the class of well-posed problems, so a small perturbation of

fuzzy parameters may cause only a small deviation in the fuzzy decision. Namely, we shall show some results connected with a stability property of the fuzzy decision in possibilistic linear equality systems.

2. Preliminaries.

A fuzzy number is a fuzzy set $\tilde{a}$ with membership function $\tilde{a} : \mathcal{R} \to I = [0,1]$ which is upper semi-continuous, normal, convex and compactly supported.
The set of all fuzzy numbers we shall denote by $\mathcal{F}$. To distinguish a fuzzy number from a crisp one, the former will always be denoted with a tilde $\tilde{}$. A symmetrical trapezoidal fuzzy number $\tilde{a}$ denoted by $\tilde{a} = (a,\alpha,\theta)$ is defined as [1]

$$\tilde{a}(t) = \begin{cases} 1 & \text{if } |a-t| \le \theta \\ 1 + \frac{\theta}{\alpha} - \frac{|a-t|}{\alpha} & \text{if } \theta < |a-t| < \theta + \alpha \\ 0 & \text{if } |a-t| \ge \theta + \alpha \end{cases}$$

where a is the center, $\theta > 0$ is the upper width and $\alpha > 0$ is the lower width of fuzzy number.
Consider the following extremal cases:

(i) If $\alpha > 0$ but $\theta = 0$ we get a symmetrical triangular fuzzy number with center a and width α denoted by $\tilde{a} \doteq (a,\alpha)$.

(ii) If $\alpha = 0$ but $\theta > 0$ we get a crisp flat fuzzy number with center a and width θ denoted by $\tilde{a} = [a,\theta]$.

(iii) If $\alpha = 0$ and $\theta = 0$ we get a crisp fuzzy number with support $\{a\}$.

Let $L > 0$ be a real number. By $F(L)$ we denote the set of all fuzzy numbers $\tilde{a}$ with membership function satisfying the Lipschitz condition with constant L, i.e.

$$|\tilde{a}(t) - \tilde{a}(t')| \leq L|t-t'| \quad \text{for all} \quad t,t' \in R$$

The truth value of the assertion "$\tilde{a}$ equal to $\tilde{b}$", which we write $\tilde{a} \doteq \tilde{b}$ is Poss $(\tilde{a} \doteq \tilde{b})$ defined as

$$\text{Poss}(\tilde{a} \doteq \tilde{b}) = \sup_{t \in R} \tilde{a}(t) \wedge \tilde{b}(t)$$

We define a metric D in F by the equation

$$D(\tilde{a},\tilde{b}) = \sup_{\alpha \in I} h([\tilde{a}]^{\alpha},[\tilde{b}]^{\alpha}) \tag{2}$$

where $[\tilde{a}]^{\alpha}$, $[\tilde{b}]^{\alpha}$ denote the α-level sets of fuzzy numbers $\tilde{a},\tilde{b} \in F$ and h is the Hausdorff metric on compact subsets of R^2.

3. Possibilistic Linear Equality Systems.

Generalizing the system (1) consider the following possibilistic linear system

$$\tilde{a}_{i1}x + \ldots + \tilde{a}_{in}x_n \doteq \tilde{b}_i, \quad i = \overline{1,m} \tag{3}$$

where $\tilde{a}_{ij}$, $\tilde{b}_i$ are fuzzy numbers and $\doteq$ denotes possibility. We denote by $\mu_i(x)$ the degree of satisfaction i's equation at the point $x \in R^n$ in (3), i.e.

$$\mu_i(x) = \mathrm{Poss}(\tilde{a}_{i1}x_1 + \ldots + \tilde{a}_{in}x_n \tilde{=} \tilde{b}_i)$$

Then the fuzzy decision of the system (3) is defined by

$$\mu(x) = \min_{i=\overline{1,m}} \mu_i(x) \tag{4}$$

A measure of consistency for the system (3) can be [2]

$$\mu^* = \mu^*(x^*) = \sup_{x \in R^n} \mu(x) \tag{5}$$

and x^* is a maximizing solution.

In many important cases the fuzzy parameters $\tilde{a}_{ij}$, $\tilde{b}_i$ of the system (3) are not known exactly [3] and we have to work with their approximations $\tilde{a}^\delta_{ij}$, $\tilde{b}^\delta_i$ such that

$$\max_{i,j} D(\tilde{a}_{ij}, \tilde{a}^\delta_{ij}) \le \delta, \ \max_i D(\tilde{b}_i, \tilde{b}^\delta_i) \le \delta \tag{6}$$

where the metric D is defined by (2).

Then we get the following system with perturbed fuzzy parameters

$$\tilde{a}^\delta_{i1}x_1 + \ldots + \tilde{a}^\delta_{in}x_n \tilde{=} \tilde{b}^\delta_i, \qquad i = \overline{1,m} \tag{7}$$

In a similar manner we can define the fuzzy decision of the perturbed system (7)

$$\mu^\delta(x) = \min_{i=\overline{1,m}} \mu^\delta_i(x)$$

and the maximizing solution $x^*(\delta)$ satisfies the equation

$$\mu^*(\delta) = \mu(x^*(\delta)) = \sup_{x\in R^n} \mu^\delta(x)$$

Next we shall outline some results connected with a stability property of fuzzy decision (and measure of consistency) of possibilistic linear equality systems (3) with respect to changes of fuzzy parameters in metric D.

4. Results.

Theorem 1. Let $L > 0$ be a real number and $\tilde{a}_{ij}$, $\tilde{a}^\delta_{ij}$, $\tilde{b}_i$, $\tilde{b}^\delta_i \in F(L)$ are fuzzy numbers. If (6) hold, then

(i) $\| \mu-\mu^\delta \|_C = \sup_{x\in R^n} |\mu(x)-\mu^\delta(x)| \le 2L\delta$

(ii) $|\mu^*-\mu^*(\delta)| \le 2L\delta$

where μ, μ^δ are the fuzzy decisions and μ^*, $\mu^*(\delta)$ are the measures of consistency for the systems (3), (7) respectively.

Let us consider the system (3) with symmetrical trapezoidal fuzzy numbers $\tilde{a}_{ij} = (a_{ij}, \alpha, \Theta)$, $\tilde{b}_i = (b_i, \alpha, \Theta)$:

$$(a_{i1},\alpha,\Theta)x_1 + \ldots + (a_{in},\alpha,\Theta)x_n \cong (b_i,\alpha,\Theta), \quad i = \overline{1,m} \tag{8}$$

and consider the perturbed system (7) with $\tilde{a}^\delta_{ij} = (a_{ij}(\delta), \alpha,\Theta)$, $\tilde{b}^\delta_i = (b_i(\delta), \alpha,\Theta)$ (i.e. perturbations are allowed only in the centres of fuzzy parameters):

$$(a_{i1}(\delta),\alpha,\Theta)x_1 + \ldots + (a_{in}(\delta),\alpha,\Theta)x_n \cong (b_i(\delta),\alpha,\Theta), \quad i=\overline{1,m} \tag{9}$$

Then the fuzzy decision (4) of the system (8) is [1]

$$\mu(x) = \begin{cases} 1 & \text{if } |Ax-b|_\infty \le \theta(|x|_1+1), \\ 1+\frac{\theta}{\alpha}-\frac{|Ax-b|_\infty}{\alpha(|x|_1+1)} & \text{otherwise,} \\ 0 & \text{if } |Ax-b|_\infty \ge (\theta+\alpha)(|x|_1+1), \end{cases} \tag{10}$$

where $A=(a_{ij})$, $b=(b_i)$, $|x|_\infty = \max\{|x_i|,\ i=\overline{1,m}\}$, $|x|_1 = |x_1|+\ldots+|x_n|$.

As a consequence of theorem 1 we obtain the following result

Theorem 2. [1] Let $\mu(x)$ and $\mu^\delta(x)$ be the fuzzy decisions to the systems (8), (9) respectively. If

$$\max_{i,j}|a_{ij}-a_{ij}(\delta)| \le \delta, \quad \max_i|b_i-b_i(\delta)| \le \delta, \tag{11}$$

then

(i) $$\|\mu-\mu^\delta\|_C \le \frac{\delta}{\alpha} \tag{12}$$

(ii) $$|\mu^*-\mu^*(\delta)| \le \frac{\delta}{\alpha} \tag{13}$$

where μ^* and $\mu^*(\delta)$ are the measures of consistency for the systems (8), (9) respectively.

Remark 1. From (12) and (13) it follows that theorem 2 is valid when $\theta=0$, but $\alpha>0$, i.e. when the fuzzy parameters of the systems (8), (9) are of symmetrical triangular form [4].

Remark 2. From (10), (12) it follows that the fuzzy decision of the system (8) can be regarded as a stable generalized solution (in classical sense) to the system (1) (with respect to changes of non-fuzzy parameters by (11)).

Remark 3. The system (8) with crisp flat fuzzy numbers may be instable with respect to changes of centers of fuzzy numbers, e.g.

Consider the following original system

$$[0,1]x \tilde{=} [2,1] \tag{14}$$

and the perturbed system

$$[\delta,1]x \tilde{=} [2,1] \tag{15}$$

Then we get

$$\mu^{\delta}\left(\frac{1}{1+\delta}\right) = 1, \quad \mu\left(\frac{1}{1+\delta}\right) = 0 \quad \text{for all} \quad 0 < \delta \leq 1$$

So

$$1 = \| \mu - \mu^{\delta} \|_C \not\to 0 \quad \text{if} \quad \delta \to 0$$

this means the instability of the fuzzy decision in the problems (14), (15) (in metric C).

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