SOME THEOREMS ON LIMIT OF FUZZY NUMBERS

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Abstract

In this paper, we'll discuss the problem of the existence of the limit of the sequence of fuzzy numbers and give some important theorems of fuzzy numbers: monotone convergence theorem, nested intervals theorem, Cauchy criterion for convergence, accumulation principle, etc.

Keywords: Fuzzy number, Fuzzy distance.

[1] has first introduced an important concept of the fuzzy distance of two fuzzy numbers, and given a concrete fuzzy distance, which possesses almost every property of the absolute value of real numbers, and defined the limit of the sequence of fuzzy numbers in the fuzzy distance, and obtained some results similar to those of the limit of the sequence of real numbers. In the paper, we'll give some important theorems on the existence of the limit of the sequence of fuzzy numbers.

Let A^* be a class of some subsets of $F^*(R)$[1], and A^* has the properties:

1) Whenever $A \in A^*$, if A is upper bounded, then there exists $\sup A \in F^*(R)$;

2) Whenever $A \in A^*$, if A is lower bounded, then there exists $\inf A \in F^*(R)$.

Obviously, A^* is nonempty.

<u>Definition 1.</u> Let $\underline{a}$, $\underline{b} \in F^*(R)$, $\underline{a} \leq \underline{b}$, we define

$$[\underline{a}\ ,\ \underline{b}] \triangleq \{\underline{x};\ \underline{a} \leq \underline{x} \leq \underline{b},\ \underline{x} \in F^*(R)\},$$

then $[\underline{a}\ ,\ \underline{b}]$ is said to be a closed interval.

Similarly, we introduce

<u>Definition 2.</u> Let $\underline{a}$, $\underline{b} \in F^*(R)$, $\underline{a} < \underline{b}$, we define

$$(\underline{a}\ ,\ \underline{b}) \triangleq \{\underline{x};\ \underline{a} < \underline{x} < \underline{b},\ \underline{x} \in F^*(R)\},$$

then $(\underline{a}\ ,\ \underline{b})$ is said to be an open interval.

<u>Definition 3.</u> Let $\{\underline{a}_n\} \subset F^*(R)$, if

$$\underline{a}_n \leq \underline{a}_{n+1},$$

for every n, then $\{\underline{a}_n\}$ is said to be a monotone increasing sequence of fuzzy numbers. If

$$\underline{a}_{n+1} \leq \underline{a}_n,$$

for every n, then $\{\underline{a}_n\}$ is said to be a monotone decreasing sequence of fuzzy numbers.

<u>Theorem 1.</u> (Monotone convergence theorem) Let $\{\underline{a}_n\} \in A^*$,

1) If $\underline{a}_n$ is a monotone increasing sequence, and has upper bound $\underline{M}$ ($\neq \underline{\infty}$) $\in F^*(R)$, then $\{\underline{a}_n\}$ is convergent, and

$$\lim_{n\to\infty} \underline{a}_n = \sup_{n\geq 1} \{\underline{a}_n\};$$

2) If $\{\underline{a}_n\}$ is a monotone decreasing sequence, and has

lower bound $\underline{m}$ ($\neq \underline{\infty}$) $\in$ $\mathbf{F}^*(R)$, then $\{\underline{a}_n\}$ is convergent, and

$$\lim_{n\to\infty} \underline{a}_n = \inf_{n\geq 1} \{\underline{a}_n\}.$$

Proof. 1) For arbitrary given $\varepsilon > 0$, by using definition of the least upper bound, there exista an integer $N > 0$ such that

$$\sup_{n\geq 1} \{\underline{a}_n\} < \underline{a}_N + \varepsilon.$$

Since $\{\underline{a}_n\}$ is a monotone increasing sequence of fuzzy numbers, then

$$\underline{a}_n \leq \sup_{n\geq 1} \{\underline{a}_n\} < \underline{a}_N + \varepsilon \lneq \underline{a}_n + \varepsilon,$$

as $n \geq N$, therefore, when $n \geq N$, we have

$$\underline{\rho}(\underline{a}_n , \sup_{n\geq 1} \{\underline{a}_n\}) \leq \underline{\rho}(\underline{a}_n , \underline{a}_n + \varepsilon) = \underline{\rho}(\varepsilon , 0) = \varepsilon.$$

That is to say

$$\lim_{n\to\infty} \underline{a}_n = \sup_{n\geq 1} \{\underline{a}_n\}.$$

2) Similar.

Theorem 2. (Nested intervals theorem) Let $\{[\underline{a}_n , \underline{b}_n]\}$ is a closed intervals sequence of A*, if it has the properties:

1) $\underline{a}_n \leq \underline{a}_{n+1} < \underline{b}_{n+1} \leq \underline{b}_n$, $n = 1, 2, \cdots \cdots$, $\underline{a}_1$, $\underline{b}_1 \neq \underline{\infty}$;

2) $\underline{\rho}(\underline{a}_n , \underline{b}_n) \longrightarrow 0$ $(n \longrightarrow \infty)$,

then

$$\lim_{n\to\infty} \underline{a}_n = \lim_{n\to\infty} \underline{b}_n \triangleq \underline{a},$$

and $\underline{a}$ is uniquely common point of the closed intervals.

Proof. By hypothesis of theorem, we know

$$\underline{a}_1 \leq \underline{a}_2 \leq \cdots \cdots \leq \underline{a}_n = \cdots \cdots \leq \underline{b}_1;$$

$$\underline{b}_1 \geq \underline{b}_2 \geq \cdots \cdots \geq \underline{b}_n \geq \cdots \cdots \geq \underline{a}_1,$$

therefore $\{\underline{a}_n\}$ is a monotone increasing sequence, and has upper bound $\underline{b}_1$,,$\{\underline{b}_n\}$ is a monotone decreasing sequence, and has lower bound $\underline{a}_1$. It follows, by using theorem 1 that

$$\lim_{n\to\infty} \underline{a}_n = \sup_{n\geq 1} \{\underline{a}_n\};$$

$$\lim_{n\to\infty} \underline{b}_n = \inf_{n\geq 1} \{\underline{b}_n\}.$$

Consequenlty, we have

$$\underline{a}_k \leq \lim_{n\to\infty} \underline{a}_n \quad \text{and} \quad \lim_{n\to\infty} \underline{b}_n \leq \underline{b}_k,$$

for every integer $k \succ 0$, from theorem 3.6[1]

$$\underline{a}_k \leq \lim_{n\to\infty} \underline{a}_n \leq \underline{b}_k, \quad \underline{a}_k \leq \lim_{n\to\infty} \underline{b}_n \leq \underline{b}_k, \qquad (*)$$

for every integer $k \succ 0$. Therefore, we have

$$\underline{\rho}(\lim_{n\to\infty} \underline{a}_n , \lim_{n\to\infty} \underline{b}_n) = 2\cdot\underline{\rho}(\underline{a}_k , \underline{b}_k),$$

for every integer $k \succ 0$, it follows that

$$0 \leq \underline{\rho}(\lim_{n\to\infty} \underline{a}_n , \lim_{n\to\infty} \underline{b}_n) = 2\lim_{k\to\infty} \underline{\rho}(\underline{a}_k , \underline{b}_k) = 0,$$

that is to say

$$\lim_{n\to\infty} \underline{a}_n = \lim_{n\to\infty} \underline{b}_n = \underline{a}.$$

Since $\underline{a} = \sup_{n\geq 1} \{\underline{a}_n\} = \inf_{n\geq 1} \{\underline{b}_n\}$, we have

$$\underline{a} \in [\underline{a}_n , \underline{b}_n],$$

for every integer n.

Suppose also that there exists $\underline{a}'$ with

$$\underline{a}' \in [\underline{a}_n , \underline{b}_n],$$

for every integer n, it follows, by using theorem 2.2.6)[1] that

$$\rho(\underline{a} , \underline{a}') \leq 2\cdot\rho(\underline{a}_n , \underline{b}_n),$$

for every integer n. Therefore

$$0 \leq \rho(\underline{a} , \underline{a}') = 2\lim_{n\to\infty} \rho(\underline{a}_n , \underline{b}_n) = 0,$$

thus

$$\rho(\underline{a} , \underline{a}') = 0 \quad \text{or} \quad \underline{a} = \underline{a}'.$$

<u>Definition 4.</u> Let $\{\underline{a}_n\} \subset F^*(R)$, and has bounded, we define

$$\overline{\lim_{n\to\infty}}\, \underline{a}_n = \lim_{k\to\infty} \sup_{n\geq k} \{\underline{a}_n\};$$

$$\underline{\lim_{n\to\infty}}\, \underline{a}_n = \lim_{k\to\infty} \inf_{n\geq k} \{\underline{a}_n\},$$

then $\overline{\lim\limits_{n\to\infty}}\, \underline{a}_n$, $\underline{\lim\limits_{n\to\infty}}\, \underline{a}_n$ are said to be the upper limit and the lower limit of $\{\underline{a}_n\}$.

<u>Theorem 3.</u> Let $\{\underline{a}_n\} \in A^*$, $\underline{a}$ $F^*(R)$, then $\lim\limits_{n\to\infty} \underline{a}_n = \underline{a}$ if and only if $\overline{\lim\limits_{n\to\infty}}\, \underline{a}_n = \underline{\lim\limits_{n\to\infty}}\, \underline{a}_n = \underline{a}$.

Proof. Necessity. Obvious.

Sufficiency. For any $\varepsilon > 0$, since $\overline{\lim\limits_{n\to\infty}}\, \underline{a}_n = \underline{\lim\limits_{n\to\infty}}\, \underline{a}_n = \underline{a}$, we can always find two integers $K_1, K_2 > 0$ such that

$$\rho(\sup_{n\geq k} \{\underline{a}_n\} , \underline{a}) < \tfrac{1}{2}\varepsilon ,$$

as $k \geq K_1$, and

$$\rho(\inf_{n\geq k} \{\underline{a}_n\} , \underline{a}) < \tfrac{1}{2}\varepsilon ,$$

as $k \geq K_2$. Let $K = \max(K_1 , K_2)$, when $k \geq K$, we have

$$\rho(\inf_{n\geq k} \{\underline{a}_n\} , \sup_{n\geq k} \{\underline{a}_n\}) < \varepsilon .$$

On the other hand, for every k, we have

$$\inf_{n\geq k} \{\underline{a}_n\} \leq \underline{a}_k \leq \sup_{n\geq k} \{\underline{a}_n\},$$

thus, when $k \geq K$,

$$\rho(\underline{a}_k , \inf_{n\geq k} \{\underline{a}_n\}) < \varepsilon ,$$

it results that

$$\rho(\underline{a}_k , \underline{a}) = \rho(\underline{a}_k , \inf_{n\geq k} \{\underline{a}_n\}) + \rho(\inf_{n\geq k} \{\underline{a}_n\} , \underline{a})$$

$$< \varepsilon + \tfrac{1}{2}\varepsilon = \tfrac{3}{2}\varepsilon ,$$

as $k \geq K$, that is to say

$$\lim_{n\to\infty} \underline{a}_n = \underline{a}.$$

<u>Definition 5.</u> Let $\{\underline{a}_n\} \subset F^*(R)$, then $\{\underline{a}_n\}$ is called a fundamental sequence, if for any $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$\rho(\underline{a}_n , \underline{a}_m) < \varepsilon ,$$

as $n, m \geq N$.

<u>Theorem 4.</u> (Cauchy criterion for convergence) Let $\{\underline{a}_n\} \in A^*$, then $\{\underline{a}_n\}$ is convergent if and only if $\{\underline{a}_n\}$ is the fundamental sequence.

Proof. Necessity. Obvious.

Sufficiency. For any $\varepsilon > 0$, let $m = N + 1$, then

$$\rho(\underline{a}_n , \underline{a}_{N+1}) < \varepsilon ,$$

as $n \geq N$. Therefore

$$\underline{a}_{N+1} - \varepsilon \leq \underline{a}_n \leq \underline{a}_{N+1} + \varepsilon ,$$

as $n \geq N$, hence

$$\underline{a}_{N+1} - \varepsilon \leq \inf_{k\geq n} \{\underline{a}_k\} \leq \sup_{k\geq n} \{\underline{a}_k\} \leq \underline{a}_{N+1} + \varepsilon ,$$

as $n \geq N$. By using theorem 1 and definition 4 that

$$\underline{a}_{N+1} - \varepsilon \leq \underline{\lim}_{n\to\infty} \underline{a}_n \leq \overline{\lim}_{n\to\infty} \underline{a}_n \leq \underline{a}_{N+1} + \varepsilon ,$$

this shows that

$$\rho(\underline{\lim}_{n\to\infty} \underline{a}_n , \overline{\lim}_{n\to\infty} \underline{a}_n) \leq 2\varepsilon .$$

Consequently,

$$\underline{\lim}_{n\to\infty} \underline{a}_n = \overline{\lim}_{n\to\infty} \underline{a}_n = \underline{a} \in F^*(R),$$

from theorem 3

$$\lim_{n\to\infty} \underline{a}_n = \underline{a}$$

Definition 6. Let $A \subset F^*(R)$, $\mathcal{E}$ be a nonempty class of open intervals of $F^*(R)$, we say that A is covered by $\mathcal{E}$, if for every $\underline{a} \in A$, there exists at least an open interval $\underline{Q}$ ($\underline{Q} \in \mathcal{E}$) such that $\underline{a} \in \underline{Q}$.

Definition 7. Let $A \subset F^*(R)$, $\underline{a}$, $\underline{b} \in A$ and $\underline{a} \leq \underline{b}$, A is called a M-closed interval, write $[\underline{a} , \underline{b}]^*$, if for every $\underline{c}$, $\underline{d} \in A$, with the following properties:

1) $\underline{a} \leq \underline{c} \leq \underline{d} \leq \underline{b}$;

2) $(\underline{c} + \underline{d})/2 \in A$.

Theorem 5. (Finite covering theorem) Let $[\underline{a} , \underline{b}]^* \in A^*$, $\underline{a}$, $\underline{b} \neq \underline{\infty}$, if $[\underline{a} , \underline{b}]^*$ can be covered by a class $\mathcal{E}$ of open intervals, then there exist finite open intervals $\underline{Q}_i \in \mathcal{E}$, $i = 1, 2, \cdots\cdots, n$ such that $[\underline{a} , \underline{b}]^*$ can be covered by $\mathcal{E}' = \{Q_i;\ i = 1, 2, \cdots\cdots, n\}$.

Proof. Suppose $[\underline{a} , \underline{b}]^*$ cannot be covered by finite open intervals of $\mathcal{E}$, then there exists at least a closed interval $[\underline{a} , \frac{\underline{a}+\underline{b}}{2}]^*$ or $[\frac{\underline{a}+\underline{b}}{2} , \underline{b}]^*$ which cannot be covered by finite open intervals of $\mathcal{E}$. There is no harm, suppose $[\underline{a} , (\underline{a}+\underline{b})/2]^*$ cannot be covered by finite open intervals of $\mathcal{E}$, and we have

$$\underline{\rho}(\underline{a}\ ,\ (\underline{a}+\underline{b})/2) = \tfrac{1}{2}\underline{\rho}(\underline{a}\ ,\ \underline{b}).$$

Let $\underline{a}_1 = \underline{a}$, $\underline{b}_1 = (\underline{a}+\underline{b})/2$, that is to say $[\underline{a}_1\ ,\ \underline{b}_1]^*$ cannot be covered by finite open intervals of $\mathcal{E}$, thus there exists at least a closed interval $[\underline{a}_1\ ,\ (\underline{a}_1+\underline{b}_1)/2]^*$ or $[(\underline{a}_1+\underline{b}_1)/2\ ,\ \underline{b}_1]^*$ which cannot be covered by finite open intervals of $\mathcal{E}$, we write $[\underline{a}_2\ ,\ \underline{b}_2]^*$ and have

$$\underline{\rho}(\underline{a}_2\ ,\ \underline{b}_2) = \tfrac{1}{4}\underline{\rho}(\underline{a}\ ,\ \underline{b}).$$

Repeating the procedure step by step, we obtain $\{[\underline{a}_n\ ,\ \underline{b}_n]^*\}$ with the properties:

1) $\underline{a} \leq \underline{a}_1 \leq \underline{a}_2 \leq \cdots \leq \underline{a}_n \leq \cdots \leq \underline{b}_n \leq \cdots \leq \underline{b}_2 \leq \underline{b}_1 \leq \underline{b}$;

2) $\underline{\rho}(\underline{a}_n\ ,\ \underline{b}_n) = \frac{1}{2^n}\underline{\rho}(\underline{a}\ ,\ \underline{b})$,

$$\lim_{n\to\infty} \underline{\rho}(\underline{a}_n\ ,\ \underline{b}_n) = 0,$$

therefore, by using theorem 2 that there exists unique $\underline{c} \in [\underline{a}_n\ ,\ \underline{b}_n]^*$ $n = 1, 2, \cdots$, according to the definition of $[\underline{a}_n\ ,\ \underline{b}_n]^*$, $\underline{c}$ cannot be covered by finite open intervals of $\mathcal{E}$. On the other hand, since $\underline{c} \in [\underline{a}\ ,\ \underline{b}]^*$ and $[\underline{a}\ ,\ \underline{b}]^*$ can be covered by $\mathcal{E}$, then there exists at least an open interval $\underline{O} \in \mathcal{E}$ such that $\underline{c} \in \underline{O}$. This shows that $[\underline{a}\ ,\ \underline{b}]^*$ can be covered by finite open intervals of $\mathcal{E}$, and compete the proof of the theorem.

Definition 8. Let $A \subset F^*(R)$, $\underline{a} \in F^*(R)$, A contains infinite fuzzy numbers, then $\underline{a}$ is said to be an accumulation point of A, if for any $\varepsilon > 0$, $(\underline{a}-\varepsilon\ ,\ \underline{a}+\varepsilon)$ contains infinite fuzzy numbers of A.

Theorem 6. (Accumulation principle) If $A \subset [\underline{a}, \underline{b}]^*$ and A contains infinite fuzzy numbers, then there exists at least an accumulation point of A.

Proof. Obvious.

Corollary. If $\{\underline{a}_n\} \subset [\underline{a}, \underline{b}]^*$, then there exists a subsequence $\{\underline{a}_{n_i}\}$ of $\{\underline{a}_n\}$ such that $\{\underline{a}_{n_i}\}$ is convergent.

Proof. Obvious.

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