SYMMETRIC FUZZY PROBABILITY MEASURES AND THE BAYES FORMULA

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Abstract: We study a symmetric fuzzy probability measure m, i.e. a fuzzy probability in sense of Klement & all [2] possessing the complement property $m(\mu) + m(\mu') = 1$. We generalize the results of Piasecki [4,5,7] on the Bayes formula for fuzzy probability measures. Our generalization is built on the notion of m-disjointness. We study σ_m the smallest soft algebra containing all m-Bayes fuzzy partitions for a symmetric fuzzy probability measure m. The limit cases lead to the crisp case, resp. to the Piasecki's concepts.

1. Symmetric fuzzy probability measures

Originally a fuzzy probability was introduced by Zadeh in [8]. He started with a classical probability space $(X, \mathcal{L}, P)$ and he defined a fuzzy probability m by the following formula

$$m(\mu) = \int_X \mu \, dP \quad , \quad \mu \in F(\mathcal{L}) \tag{1.1}$$

Here a fuzzy subset A in X is described by its membership function $\mu: X \longrightarrow [0\ ,\ 1]$. $F(\mathcal{L})$ denotes the family of all $\mathcal{L}$-measurables fuzzy subsets in X, which will be called a generated fuzzy σ-algebra.

Klement, Lowen and Schwychla [2] define a fuzzy probability measure on a fuzzy σ-algebra σ as the continuous from below mapping $m: \sigma \longrightarrow [0\ ,\ 1]$ fulfilling the next properties:

$$m(0_X) = 0 \tag{1.2}$$

$$m(1_X) = 1 \tag{1.3}$$

$$m(\mu \cup \nu) + m(\mu \cap \nu) = m(\mu) + m(\nu) \tag{1.4}$$

for each $\mu, \nu \in \sigma$.

Here $\cup$, $\cap$ are the fuzzy connectives of the fuzzy union and the fuzzy intersection.

It is easy to see that (1.1) defines really a fuzzy probability in the sense of [2] for an arbitrary choice of fuzzy union and intersection preserving the valuation property. Moreover, if we use the classical complementation $\mu' = 1 - \mu$ (1.5), then (1.1) implies

$$m(\mu') = 1 - m(\mu) \qquad \forall \mu \in F(\mathcal{L}) \tag{1.6}.$$

Definition 1.1. A fuzzy probability measure fulfilling (1.6) on σ will be called symmetric fuzzy probability measure.

In what follows we will deal with the fuzzy connectives of union and intersection, which are commutative, associative, distributive, continuous and increasing in both places, satisfying

$$\mu \cap 1_X = \mu,\ \mu \cup 0_X = \mu,\ \mu \cap \nu \subset \mu,\ \mu \cup \nu \supset \mu,\ \mu, \nu \in \sigma \tag{1.7}.$$

Bellman and Giertz [1] have shown that all these properties are satisfied only by the classical Zadeh's fuzzy connectives

$$\mu \cap \nu = \min(\mu, \nu) \tag{1.8}$$
$$\mu \cup \nu = \max(\mu, \nu) \tag{1.9}.$$

To preserve the De Morgan laws we can take arbitrary complementation connective induced by the strong negations. In what follows wd will use the classical Zadeh's complementation (1.5).

The problem of integral reprezentation of a general fuzzy probability on a generated fuzzy σ-algebra $\sigma = F(\mathcal{L})$ was solwed by Klement in [3].

Theorem 1.1. (Klement, [3]) Let m be a fuzzy probability measure on $F(\mathcal{L})$. Then there exist one and only one probability P on $(X, \mathcal{L})$ and a P-almost everywhere uniquely determined Markoff-kernel K such that

$$\forall \mu \in F(\mathcal{L}):\ m(\mu) = \int_X K(x, [0, \mu(x)[)\, dP(x) \tag{1.10}.$$

Recall that a Markoff-kernel is a function

$$K: X \times \mathcal{B}_{[0,1[} \longrightarrow [0, 1] \tag{1.11}$$

such that following two conditions are fulfilled:

$\forall B \in \mathcal{B}_{[0\,,\,1[}$: $K(\,.\,,\,B\,): X \longrightarrow [0\,,\,1]$ is $\mathcal{A}$-$\mathcal{B}$measurable (1.12)

$\forall x \in X$: $K(\,x,\,.\,): \mathcal{B}_{[0\,,\,1[} \longrightarrow [0\,,\,1]$ is a probability (1.13).

If for P-a.e. $x \in X$ the probabilities (1.13) are continuous, i.e. they can be described by the density functions $k(\,x,\,.\,): [0\,,\,1[\longrightarrow R^+$, then the kernel is continuous.

Theorem 1.2. m is a symmetric fuzzy probability measure on $F(\mathcal{A})$ iff m is of type (1.10) for a symmetric Markoff-kernel K, i.e. for a continuous Markoff-kernel K satisfying the following property:

for P-a.e. $x \in X, \forall y \in]0\,,\,1[$:

$$K(\,x,[0\,,\,y[) + K(\,x,[0\,,\,1-y[) = 1 \qquad (1.14).$$

The property (1.14) is equivalent to the following one:
for P-a.e. $x \in X, \forall y \in]0\,,\,1[$: $k(\,x,\,y) = k(\,x,\,1-y)$,

$$K(\,x,[0\,,\,1/2[\,) = 1/2 \qquad (1.15).$$

The only problem in proving the Theorem 1.2. is to show that the symmetricity of m implies (1.14) or (1.15).
So let m be a symmetric fuzzy probability measure on $F(\mathcal{A})$.
Denote by A the crisp subset of all $x \in X$, for which there is a $y_x \in]0\,,\,1[$ such that

$$K(\,x,[0\,,\,y_x[) + K(\,x,[0\,,\,1-y_x[) < 1 \qquad (1.16).$$

Define a fuzzy subset μ as

$$\mu(x) = 0\,,\ x \notin A\,,\ \mu(x) = y_x\,,\ x \in A \qquad (1.17).$$

Then

$$1 = m(\mu) + m(\mu') = P(A') + \int_A K(\,x,[0\,,\,y_x[) + \qquad (1.18)$$
$$+ K(\,x,[0\,,\,1-y_x[)\ dP(x)\,,$$

which implies $P(A) = 0$.
Similarly we get $P(B) = 0$, where

$$B = \{x \in X, \exists\, y_x \in]0\,,\,1[\,,\ K(\,x,[0\,,\,y_x[) + K(\,x,[0\,,\,1-y_x[) > 1\,\} \qquad (1.19).$$

Example 1.1. The Zadeh's fuzzy probability m defined by (1.1) is a symmetric fuzzy probability measure,

$$\forall x \in X, \forall y \in [0\,,\,1[: K(\,x, [0\,,\,y[\,) = y\,,\; k(\,x,\,y) = 1 \qquad (1.20)$$

In what follows we will identify a symmetric fuzzy probability measure m with a probability P and a symmetric Markoff-kernel K, even if fuzzy σ-algebra σ is not a generated fuzzy σ-algebra, resp. σ is only algebra. Farther we allow a slight modification of the notion of symmetric Markoff-kernel K according to the general fuzzy σ-algebra (fuzzy algebra) σ. In this case we admit $K(\,x,\,.\,) = \delta_{1/2}(\,.\,)$ (the Dirac distribution concentrated in 1/2) for all $x \in C$, $C \in \mathcal{L}$, such that for all $\mu \in \sigma$, $P(D_\mu) = 0$, where

$$D_\mu = \{x \in C, \mu(x) = 1/2\}\,.$$

2. On the Bayes formula

The mapping $m(\,.\,/\mu): \sigma \longrightarrow [0\,,\,1]$ defined, for any fuzzy probability measure m and for each $\mu \in \sigma$ such that $m(\mu) \neq 0$, by the identity

$$m(\gamma\,/\mu) = \frac{m(\gamma \cap \mu)}{m(\mu)} \quad , \; \gamma \in \sigma \qquad (2.1)$$

is called a conditional fuzzy probability given μ.

Piasecki in [4] has defined a Bayes fuzzy partition (briefly BFP) of X as the system (finite or countable) $\{\mu_i\}_{i \in I}$, $\mu_i \in \sigma$, of fuzzy subsets satisfying the following conditions:

(R1) the fuzzy subsets $\{\mu_i\}$ are pairwise W-separated, i.e.

$$\mu_i \leq 1 - \mu_j = \mu_j' \quad \text{for every } i \neq j,\; i,\; j \in I$$

(R2) $m(\sup_{i \in I} \mu_i) = 1$

(R3) $m(\mu_i) > 0$ for each $i \in I$.

A system $\{\mu_i\}$ of fuzzy subsets satisfying (R1), (R2) and (R3) will be called W-Bayes fuzzy partition (W-BFP) .

Piasecki and Świtalski in [7] have proposed another concept of fuzzy disjointness:

(R1a) the fuzzy subsets $\{\mu_i\}$ are pairwise F-separated, i.e.

$$\mu_i \cap \mu_j \subset (1/2)_X \quad \text{for every } i \neq j,\ i,\ j \in I \ .$$

A system $\{\mu_i\}$ of fuzzy subsets satisfying (R1a), (R2) and (R3) will be called F-Bayes fuzzy partition (F-BFP) .

We recall some of Piasecki's definitions and results.

<u>Definition 2.1.</u> Let σ be a fuzzy algebra (fuzzy σ-algebra). If it does not contain the fuzzy subset $(1/2)_X$ then it is called soft fuzzy algebra (soft fuzzy σ-algebra).

<u>Definition 2.2.</u> Let σ be a soft fuzzy algebra (soft fuzzy σ-algebra). A fuzzy P-measure is a mapping $p\colon \sigma \to [0\ ,\ 1]$ such that:

for any $\mu \in \sigma$: $p(\mu \cup \mu') = 1$ (2.2)

if $\{\mu_i\}_{i\in I}$ fulfils (R1) and $\sup\limits_{i\in I} \mu_i \in \sigma$ then

$$p(\ \sup_{i\in I} \mu_i\ ) = \sum_{i\in I} p(\mu_i) \qquad (2.3)\ .$$

Any fuzzy P-measure is a symmetric fuzzy probability measure. Note that it may be always define by means of (1.10), where $K(\ x,\ .\ ) = \delta_{1/2}(\ .\ )$ for all $x \in X$,

$$p(\mu) = \int_X \delta_{1/2}([0\ ,\ \mu(x)[)\ dP(x) = P(\mu > 1/2\ ) \quad (2.4)\ .$$

The main results of [4, 5, 7] are expressed in the following theorem.

<u>Theorem 2.1.</u> (Piasecki, [5]) Let m be a fuzzy probability measure on a fuzzy σ-algebra σ and σ_m^W ($= \sigma_m^F$) be the smallest soft fuzzy algebra containing all W-BFP (F-BFP) generated by m. Then m satisfies the W-Bayes formula (F-Bayes formula) on σ_m^W iff it is a fuzzy P-measure on σ_m^W .

Note that $\tilde{\sigma}_m$, the smallest fuzzy σ-algebra containing all W-BFP, may be not soft, so we can deal only with $\sigma_m^W = \sigma_m^F$, which is always soft. Thus the results of [5] need to be restricted to the σ_m^W .

Definition 2.3. Let m be a given fuzzy probability measure on a fuzzy σ-algebra σ , σ_1 a fuzzy subalgebra (fuzzy sub-σ-algebra) of σ . Then m satisfies the W- (F-) Bayes formula on σ_1 iff for all $\gamma \in \sigma_1$, $m(\gamma) > 0$, and any $\{\mu_i\}_{i\in I}$ W-BFP (F-BFP) $\{\mu_i\} \in \sigma$, it holds

$$m(\mu_k/\gamma) = \frac{m(\mu_k).m(\gamma/\mu_k)}{\sum_{i\in I} m(\mu_i).m(\gamma/\mu_i)} \qquad (2.5).$$

We propose another concept of fuzzy disjointness dependent on m .

(R1b) the fuzzy subsets $\{\mu_i\}_{i\in I}$ are pairwise m-separated,

i.e. $m(\mu_i \cap \mu_j) = 0$ for every $i \neq j$, $i, j \in I$.

A system $\{\mu_i\}_{i\in I}$ of fuzzy subsets satisfying (R1b), (R2) and (R3) will be called m-Bayes fuzzy partition (m-BFP) . In [7] is defined a strong fuzzy P-measure as a measure of Definition 2.2. where the (R1) disjointness is replaced by (R1a). Similarly we can define a full fuzzy P-measure.

Definition 2.4. A fuzzy probability measure m is a full fuzzy P-measure iff it fulfils Definition 2.2., where the (R1) disjointness is replaced by (R1b).

Definition 2.5. A given fuzzy probability measure m satisfies the m-Bayes formula on σ_1 iff it fulfils Definition 2.3., where the W-BFP are replaced by m-BFP.

Lemma 2.1. Let m be a fuzzy probability measure on a fuzzy algebra (fuzzy σ-algebra) σ . Let $\mu, \gamma \in \sigma$. Then:

a) if μ, γ are W-separated, they are also F-separated
b) if m is a fuzzy P-measure and μ, γ are F-separated, they are also m-separated
c) if μ, γ are m-separated, then there exist two F-separated fuzzy subsets μ^*, γ^* such that $P(\mu \neq \mu^*) = 0$, $P(\gamma \neq \gamma^*) = 0$ (P is defined by Theorem 1.1.); here we suppose that m is a symmetric fuzzy measure.

Proof: a) is obvious, it does not depend on m.

b) Let $\mu \cap \nu \subset (1/2)_X$. Then $(\mu\cap\nu)\cup(\mu\cap\nu)' = (\mu\cap\nu)'$.

As m is a fuzzy P-measure it follows by (2.2) $m((\mu\cap\nu)') = 1$. Then (1.4) implies $m(\mu\cap\nu) = 0$, i.e. μ and ν are m-separated. It was also possible to use (2.4).

c) It is enough to prove $P(\mu\cap\nu > 1/2) = 0$. For symmetric fuzzy probability measure m it holds for all fuzzy subsets $m(\mu) \geq P(\mu > 1/2)/2$, i.e. $0 = m(\mu\cap\nu) \geq P(\mu\cap\nu > 1/2)/2$.

Note that the fuzzy P-measures and strong fuzzy P-measures are equivalent[7] . Any fuzzy P-measure is also full fuzzy P-measure but the inverse is not true.

<u>Theorem 2.2.</u> Let m be a fuzzy probability measure on a fuzzy σ-algebra σ . Then m satisfies the m-Bayes formula on σ .

Proof: The fulfilling of the m-Bayes formula on σ is equivalent to the validity of the next equation

$$\forall \nu \in \sigma: m(\nu) = \sum_{i\in I} m(\nu \cap \mu_i) \quad \text{for arbitrary m-BFP} \{\mu_i\}_{i\in I} \tag{2.6}$$

Let $\{\mu_i\}_{i\in I}$ be a fixed m-BFP, $\nu \in \sigma$. Then $\bigcup_{i\in I} \mu_i$, and the valuation property (1.4) of m implies

$$m(\nu \cap \bigcup_{i\in I} \mu_i) = m(\nu) \tag{2.7}$$

We may assume $I = \{1, 2, \ldots\}$. Then

$$\nu \cap (\mu_1 \cup \mu_2) = (\nu \cap \mu_1) \cup (\nu \cap \mu_2) \tag{2.8}$$

$$\nu \cap (\mu_1 \cap \mu_2) = (\nu \cap \mu_1) \cap (\nu \cap \mu_2) \tag{2.9}$$

and (1.4) implies

$$m(\nu \cap (\mu_1 \cup \mu_2)) + m(\nu \cap (\mu_1 \cap \mu_2)) = \\ = m(\nu \cap \mu_1) + m(\nu \cap \mu_2) \tag{2.10}$$

μ_1 and μ_2 are m-separated (R1b), $m(\mu_1 \cap \mu_2) = 0$, so that

$$m(\nu \cap (\mu_1 \cup \mu_2)) = m(\nu \cap \mu_1) + m(\nu \cap \mu_2) \tag{2.11}$$

By induction it is easy to prove that for a finite I

$$m(\nu) = m(\nu \cap \bigcup_{i\in I} \mu_i) = \sum_{i\in I} m(\nu \cap \mu_i) \text{ , q.e.d.}$$

If I is countable, we utilise the continuity from below of m.

Denote σ_m the smallest soft fuzzy algebra containing all m-Bayes fuzzy partitions. The next theorem [redacted] if we reduce our interest only to σ_m, all three concepts m-, W- and F- are equivalent for symmetric fuzzy probability measures.

Theorem 2.3. Let m be a symmetric fuzzy probability measure on a fuzzy σ-algebra σ. Then it is a full fuzzy P-measure on σ_m.

Proof: The soft fuzzy algebra σ_m consists of the elements of one-element m-BFP with theirs complements and of the elements of more numerous m-BFP.

a) Let $\mu \in \sigma_m$, $m(\mu) = 0$ or $m(\mu) = 1$. Then $m(\mu \cup \mu') = 1$, so that (2.2) is fulfilled. Now, let $\mu \in \sigma_m$, $m(\mu) \in \,]0\,,\,1[$. Then there exists a m-BFP $\{\mu_i\}_{i \in I}$, $\mu = \mu_1$. Denote

$\bigcup_{i \neq 1} \mu_i = \gamma \in \sigma$. It follows that $\{\mu, \gamma\}$ is a m-BFP, $\gamma \in \sigma_m$,

$m(\mu \cup \gamma) = 1$, $m(\mu \cap \gamma) = 0$. Let

$A = \{x \in X, \mu > \gamma\}$, $B = \{x \in X, \mu = \gamma\}$, $C = \{x \in X, \mu < \gamma\}$,

$\mu_A = \mu . I_A$ ($I_\bullet$ is the characteristic function) etc.

Then the condition (R1b) implies $m(\mu_B) = m(\mu_C) = m(\gamma_A) =$

$= m(\gamma_B) = 0$.

From (R2) it follows that $m(\mu_A) + m(\gamma_C) = 1$.

(R3) implies $m(\mu) = m(\mu_A) > 0$, $m(\gamma) = m(\gamma_C) > 0$.

It is easy to see that $P(A) = m(\mu)$, $P(B) = 0$, $P(C) = m(\gamma)$.

As m is a symmetric fuzzy probability measure, it follows

$m(\mu_A) + m(\mu'_A) = P(A)$, i.e. $m(\mu'_A) = 0$. Similarly

$m(\mu'_B) = 0$ and $m(\mu_C) = P(C)$.

Then $\{\mu, \mu'\}$ is a m-BFP and $m(\mu \cup \mu') = 1$, i.e. (2.2) is fulfilled.

b) Let $\{\mu_i\}_{i \in I}$, $\mu_i \in \sigma$, satisfies (R1b). Now, we can repeat the ideas of the proof of the Theorem 2.2. to get the equality (2.3). As $\sigma \supset \sigma_m$, m is a full fuzzy P-measure on σ_m.

Note that on σ_m the m- and F-concepts coincide P-a.e.

3. Symmetric fuzzy probability measures and the m-Bayes fuzzy partitions

In this part we study the structure of the soft fuzzy algebra σ_m. Let m be given by the probability space ($X, \mathcal{A}$, P) and a symmetric Markoff-kernel K by the formula (1.10). All the assertions below are valid in the P-a.e. sense. For the sake of simplicity we will omit "P-a.e." whenever possible.

The symmetricity of K implies that either $K(x, \cdot) = \delta_{1/2}(\cdot)$ or $K(x, \cdot)$ is a continuous probability measure,

$$\inf_{y\in]0,1]} \{y: K(x,[0,y[) = 1\} - 1/2 = 1/2 - \sup_{y\in[0,1[} \{y: K(x,[0,y[) = 0\} = \varepsilon_x \in]0 , 1/2] \quad (3.1)$$

Put $A_x = \{1/2\}$ in the first case, $A_x =]1/2 - \varepsilon_x, 1/2 + \varepsilon_x[$ in the second.

<u>Theorem 3.1.</u> Let m be a symmetric fuzzy probability measure. Then the smallest soft fuzzy algebra σ_m containing all m-BFP is equal to the algebra σ_K,

$$\sigma_K = \{\mu \in \sigma;\ P(\{x \in X, \mu(x) \in A_x\}) = 0\} \quad (3.2)$$

Proof: a) Let $\mu \in \sigma_m$. If $m(\mu) = 0$ or $m(\mu) = 1$, it is obvious that $\mu \in \sigma_K$. Let $m(\mu) \in]0 , 1[$. Then $\{\mu, \mu'\}$ forms a m-BFP. $m(\mu \cap \mu') = 0$ implies $K(x, [0 , (\mu \cap \mu')(x)[) = 0$. Similarly $m(\mu \cup \mu') = 1$ implies $K(x, [0 , (\mu \cup \mu')(x)[) = 1$. It follows $(\mu \cap \mu')(x) \notin A_x$ and $(\mu \cup \mu')(x) \notin A_x$, i.e. $\mu, \mu' \in \sigma_K$. So we have $\sigma_m \subset \sigma_K$.

b) Let $\mu \in \sigma_K$. Then $m(\mu \cap \mu') = 0$, $m(\mu \cup \mu') = 1$, so that $\{\mu, \mu'\}$ is for $m(\mu) \in]0 , 1[$ a m- BFP, $\mu, \mu' \in \sigma_m$. Therefore $\sigma_m = \sigma_K$.

<u>Corollary 3.1.</u> σ_m is a soft fuzzy σ-algebra iff

$$P(\{x \in X,\ A_x = \{1/2\}\}) = 0 \quad (3.3)$$

Remark 3.1. Let $\varepsilon_x = 1/2$ for all $x \in X$. Then $\sigma_m = \mathcal{L}$ and the only Bayes partitions are the crisp partitions.

Remark 3.2. Let $A_x = \{1/2\}$ for all $x \in X$. Then the m-principle in the Bayesian decision making is exactly the same as the F-principle and both lead to the same results as the W-principle.

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