Evaluation of fuzzy sets and fuzzy integrations : A synthetic discussion

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The different representations of a fuzzy set, in terms of a membership function or of weighted collections of crisp sets, induce various ways (which may sometimes turn to be equivalent) for extending confidence measures (probability, belief, possibility, ...) to fuzzy events, or more generally any evaluation dealing with sets (cardinality, distance, average, perimeter, ...) to fuzzy sets.

1 - Representations of a fuzzy set

A fuzzy set F (Zadeh, 1965) was originally and is usually defined by its membership function μ_F which is a function from its universe X to the real interval [0,1]. However, it was recognized soon that a fuzzy set F can be viewed as a collection of ordinary sets F_α called its α-cuts and defined by

$$\forall \alpha \in]0,1], \mu_{F_\alpha}(x) = \{x \in X, \mu_F(x) \geq \alpha\} \qquad (1)$$

Then, we have the following representation formula (Zadeh, 1971)

$$\forall x \in X, \mu_F(x) = \sup_{\alpha \in]0,1]} \min(\mu_{F_\alpha}(x), \alpha) \qquad (2)$$

Note that in (2), any operation $*$ such that $\forall \alpha \in [0,1]$, $1 * \alpha = \alpha$, and $0 * \alpha = 0$ can be used instead of min ; particularly any triangular norm (Schweizer & Sklar, 1963) can replace 'min' in (2).

Later, another representation was discovered (see Dubois and Prade, 1982a) ; namely, if the set $M = \{\alpha \in [0,1], \exists x \in X, \mu_F(x) = \alpha\}$ of membership degrees is finite, then we have

$$\forall x \in X, \mu_F(x) = \sum_{\alpha_i} \mu_{F_{\alpha_i}}(x) \cdot m_F(F_{\alpha_i}) \qquad (3)$$

where $m_F(F_{\alpha_i}) = \alpha_i - \alpha_{i+1}$ and the elements of M are decreasingly ordered : $1 = \alpha_1 > \ldots > \alpha_i > \ldots > \alpha_{n+1} = 0$. (We assume that F is normalized, i.e. $\alpha_1 = 1$ as well as its complement, i.e. $\alpha_{n+1} = 0$). Note that $\sum_i m_F(F_{\alpha_i}) = 1$. When M is no longer finite, (3) can be generalized by

$$\forall x \in X, \mu_F(x) = \int_0^1 \mu_{F_\alpha}(x) \cdot d\alpha \qquad (4)$$

2 - Extensions of a probability measure to fuzzy events

Zadeh (1968) proposed the following definition of the probability of a fuzzy event F, in terms of a Lebesgue integral, as the expectation of its membership function

$$P(F) = \int \mu_F \, . \, dP \qquad (5)$$

where P is a probability measure on a Borel field in $\mathbb{R}^n$. When X is finite, the corresponding definition is

$$P(F) = \sum_j \mu_F(x_j) \, . \, P(\{x_j\}) \qquad (6)$$

Viewing F as a weighted collection of α-cuts leads to other scalar or non-scalar definitions of the probability of a fuzzy event. First, in the representation (3), F is equivalent to the pair $(\mathcal{F}, m_F)$ with $\mathcal{F} = \{F_{\alpha_1}, \ldots, F_{\alpha_n}\}$, and then F may be viewed as a random set, i.e. F_{α_i} is a crisp realization of F with probability $m_F(F_{\alpha_i})$; m_F may also be regarded as a basic probability assignment in the sense of Shafer (1976). Then the probability of the fuzzy event F could be defined as the random number $(P(F_{\alpha_i}), m_F(F_{\alpha_i}))_{i=1,n}$; see Dubois and Jaulent (1987). A scalar counterpart of this definition is the expected value (in the sense of m_F)

$$P'(F) = \sum_i P(F_{\alpha_i}) \, . \, m_F(F_{\alpha_i}) \qquad (7)$$

in the finite case. If M is not finite, the above definition is generalized by

$$P'(F) = \int_0^1 P(F_\alpha) \, . \, d\alpha \qquad (8)$$

It is worth noticing that the two definitions (5) and (8) lead to the same evaluation, i.e. we have

$$P(F) = P'(F) \qquad (9)$$

Indeed $\int_{\mathbb{R}} \mu_F . dP = \int_{\mathbb{R}} (\int_0^1 \mu_{F_\alpha} \, . \, d\alpha) . dP = \int_0^1 (\int_{\mathbb{R}} \mu_{F_\alpha} . dP) . d\alpha$ using Fubini theorem. The equality (9) was noticed by Puri and Ralescu (1982). It is used by Andrès (1987) in a fuzzy pattern matching problem.

Second, using the representation (2), we can define a fuzzy-valued probability measure $\tilde{P}$, where $\tilde{P}(F)$ is the fuzzy set defined by

$$\forall\, r \in [0,1],\ \mu_{\tilde{P}(F)}(r) = \sup\{\eta_F(S) \mid P(S) = r\} \qquad (10)$$

with $\eta_F(S) = \inf\{\mu_F(x) \mid x \in S\}$ where S is an ordinary subset which includes the core of F (i.e. the 1-cut) and which is included in the support of F, i.e. $\{x \in X, \mu_F(x) > 0\}$; see Dubois and Prade (1982b). This definition remedies some drawbacks of a proposal by Yager (1979) where $\mu_{\tilde{P}(F)}(r)$ was defined as $\sup\{\alpha \in]0,1] \mid P(F_\alpha) = r\}$; indeed the subsets S considered in (10) are either α-cuts of S or are nested between two α-cuts. In (Dubois and Prade, 1985a), it is proved in a finite setting that P(F) (defined by (6) or (7)) and $\tilde{P}(F)$ are related in the following way

$$E(\tilde{P}(F)) = P(F) \qquad (11)$$

where E denotes the expectation

$$E(\tilde{P}(F)) = \int_{-\infty}^{+\infty} r \, . \, d\Phi(r) = \sum r_i \, .(\Phi(r_i) - \Phi(r_{i+1})) = \sum r_i \, . \, m_F(F_{\alpha_i})$$

with $\forall\, r \in [0,1],\ \Phi(r) = 1 - \max_{s > r} \mu_{\tilde{P}(F)}(s)$ (i.e. Φ is a distribution function derived from $\mu_{\tilde{P}(F)}$) and the

r_i's are the values decreasingly ordered of the $P(F_{\alpha_i})$'s. In other words, Φ is the probability distribution function associated with the above-mentioned random number (up to a complementation to 1).

3 - Extensions of other confidence measures to fuzzy events

The degree of belief in (resp. plausibility of) a fuzzy event F has been defined by Smets (1981) as the lower (resp. upper) expectation, in the sense of Dempster (1967), of its membership function, i.e.

$$\mathrm{Bel}(F) = \int_{-\infty}^{+\infty} \mu_F \cdot d\Phi^* \; ; \; \mathrm{Pl}(F) = \int_{-\infty}^{+\infty} \mu_F \cdot d\Phi_* \qquad (12)$$

with $\Phi^*(t) = \mathrm{Pl}(\{r \mid \mu_F(r) \le t\})$ and $\Phi_*(t) = \mathrm{Bel}(\{r \mid \mu_F(r) \le t\})$. Using Fubini theorem again, as in (9), it can be seen that we have

$$\mathrm{Bel}(F) = \int_0^1 \mathrm{Bel}(F_\alpha) \cdot d\alpha \; ; \; \mathrm{Pl}(F) = \int_0^1 \mathrm{Pl}(F_\alpha) \cdot d\alpha \qquad (13)$$

This can be also readily checked in the finite case using the following results due to Smets (1981)

$$\mathrm{Bel}(F) = \sum_A m(A) \cdot \min_{x \in A} \mu_F(x) \; ; \; \mathrm{Pl}(F) = \sum_A m(A) \cdot \max_{x \in A} \mu_F(x) \qquad (14)$$

where m is the basic probability assignment which defines Bel and Pl. Indeed we have

$$\sum_i \mathrm{Bel}(F_{\alpha_i}) \cdot m_F(F_{\alpha_i}) = \sum_i \Big(\sum_{A \subseteq F_{\alpha_i}} m(A)\Big) \cdot (\alpha_i - \alpha_{i+1})$$

$$= \sum_{A \subseteq F_{\alpha_1}} m(A) \cdot \alpha_1 + \sum_{i=2}^{n} \sum_{\substack{A \not\subseteq F_{\alpha_{i-1}} \\ A \subseteq F_{\alpha_i}}} m(A) \cdot \alpha_i = \sum_A m(A) \cdot (\min_{x \in A} \mu_F(x)) = \mathrm{Bel}(F)$$

since $\alpha_i = \inf_{x \in F_{\alpha_i}} \mu_F(x) = \inf_{x \in A} \mu_F(x)$ if $A \subseteq F_{\alpha_i}$ and $A \not\subseteq F_{\alpha_{i-1}}$ ■

Clearly, the extension of a set function to fuzzy sets can be contemplated in the style of (5)-(8) or of (12)-(13), as soon as a distribution function can be associated with this set function. It only requires that the set function be monotonic with respect to set-inclusion. Sugeno's "fuzzy measures" (1974), can thus be extended to fuzzy events by means of a Lebesgue integral. See Höhle (1982), Weber (1984) and Murofushi & Sugeno (1987) for works and discussions along this line. However, note that for any set function T we can use this approach under the form

$$T(F) = \int_0^1 T(F_\alpha) \cdot d\alpha \qquad (15)$$

provided that the integral exists, even if no distribution function is associated with T. This is an alternative to Sugeno (1974)'s fuzzy integral

$$\int \mu_F \circ T = \sup_{\alpha \in]0,1]} \min(\alpha, T(F_\alpha)) \qquad (16)$$

where T is supposed to be a fuzzy measure in the sense of Sugeno ; then (16) is, as such, restricted to set functions whose range is [0,1]. The extension of a possibility measure Π to fuzzy events, as proposed by Zadeh (1978), is an example of Sugeno's fuzzy integral ; namely

$$\Pi(F) = \sup_{\alpha \in]0,1]} \min(\alpha, \Pi(F_\alpha)) \qquad (17)$$

$$= \sup_{x \in X} \min(\mu_F(x), \pi(x)) \qquad \text{with } \Pi(F_\alpha) = \sup_{x \in F_\alpha} \pi(x), \text{ using (2).}$$

N.B. : $\Pi(F)$ is still equal to

$$\Pi(F) = \sup_{\alpha \in [0,1]} \min(\alpha, \mu_{CP(F\,;\,E)}(\alpha)) \qquad (18)$$

where CP(F ; E) is the compatibility (in the sense of Zadeh, 1978) of F with respect to the fuzzy set E corresponding to the possibility distribution π, i.e. $\pi = \mu_E$ which represents the available evidence ; we have $\mu_{CP(F\,;\,E)}(\alpha) = \sup\{\mu_E(x) \mid \mu_F(x) = \alpha\}$.

Then, it is interesting to contrast the definition (17) of the possibility of a fuzzy event, with the definition (12) applied to a possibility measure (since mathematically speaking a possibility measure is a particular case of plausibility function) ; see Dubois and Prade (1985b) for a detailed study of the differences between these two ways of extending possibility measures to fuzzy events.

When applied in finite cases (15) turns to be a weighted mean, while (16) is a median, which stresses the difference of nature between these two types of evaluation.

4 - Applications

The evaluation of fuzzy sets is a basic problem of which many instances are often encountered in practice. In the following, we briefly mention some of them

- Apart from confidence measures, the cardinality of a fuzzy set is a classical example of such a problem; see Dubois and Prade (1985a) for an overview. Indices related to cardinality (denoted by $|\ |$) such as Yager' measure of specificity $\int_0^1 \frac{1}{|F_\alpha|}.d\alpha$ or Higashi-Klir's measure of imprecision $\int_0^1 \log_2(|F_\alpha|).d\alpha$ are examples of integral (15).
- In (Dubois and Prade, 1987) the mean value of a fuzzy number M (a convex normalized fuzzy set of the real line with an upper semi-continuous membership function) has been introduced in terms of lower and upper expectations in the sense of Dempster. This mean value is an interval $[e_*(M),e^*(M)]$ whose bounds can be computed in practice by

$$e_*(M) = \int_0^1 \inf\{M_\alpha\}.d\alpha \quad \text{and} \quad e^*(M) = \int_0^1 \sup\{M_\alpha\}.d\alpha \qquad (19)$$

  where $\inf\{M_\alpha\}$ ans $\sup\{M_\alpha\}$ are the bounds of the α-cut of M. The compatibility CP(F ; E), introduced above, is in general a fuzzy number defined in [0,1]. It can be shown (Dubois and Prade, 1985b), that $e^*(CP(F\,;\,E))$ is the possibility (in the sense of (12)) of the fuzzy event F, while $e_*(CP(F\,;\,E))$ gives the value of the associated necessity measure.
- In data bases, we may have to evaluate for instance the average of the salaries of young people (Prade, 1986), where the salary is supposed to be precisely known for each person registered in the data base. Here, clearly 'young' is a vague predicate which delimits a fuzzy set F of people.We may use for the evaluation $\int_0^1 av(F_\alpha).d\alpha$ where $av(F_\alpha)$ is the average of salaries of people in F_α. If the

salaries are imprecisely known $av(F_\alpha)$ is a fuzzy number which may be approximated by $e_*(av(F_\alpha))$ and $e^*(av(F_\alpha))$. Then, $[\int_0^1 e_*(av(F_\alpha)).d\alpha, \int_0^1 e^*(av(F_\alpha)).d\alpha]$ gives an evaluation of the possible range of the average salary of young people. We may also apply the approach to the maximum or the minimum salary (rather than the average salary). This is then an example of finding the extremum of a function over a fuzzy domain ; see (Dubois and Prade, 1980) for a survey on this question.

- Fuzzy digital pictures offer also examples of problems of parameter evaluations, e.g. diameter, perimeter, etc... of a fuzzy region. See (Dubois and Jaulent, 1987) where scalar evaluations of the kind of (15) are used and related to previous proposals by Rosenfeld and others (e.g. Rosenfeld and Haber, 1985).
- The definitions of scalar or fuzzy-valued distances between fuzzy sets (a question on which there exists many papers and which has applications in many fields) can be also discussed along the lines sketched in this short note. In this case we have to deal with pairs of level-cuts (F_α, G_β), with possibly $\alpha \neq \beta$.
- Lastly, in criteria aggregation problems, where criteria are of unequal importance, we are led to evaluate an alternative by computing a 'measure' of the fuzzy set of goals achieved by this alternative, which is another example of fuzzy set evaluation. The measure expresses a way of weighting the goals.

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