PRODUCT FUZZY MEASURE SPACE AND FUBINI'S THEOREMS OF ABSTRACT INTEGRALS ON FUZZY SETS

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Abstract

This paper is the summary of "Product Fuzzy Measure Space and Fubini's Theorems of Abstract Integrals on Fuzzy Sets", the abstract integrals on fuzzy sets are further discussed on the basis of the results proposed in[3,4,5,6], The Cartesian product for fuzzy sets and the product fuzzy σ-algebra are introduced. The concept of the sections for fuzzy sets is given and some properties of the sections are studied. On the product fuzzy measurable space, the product fuzzy measure is defined and the Fubini's theorems of the abstract integrals on fuzzy sets are proved.

Keywords: Fuzzy measure, Abstract integral, Product fuzzy σ-algebra, Product fuzzy measure space.

§1 Abstract Integrals on Fuzzy Sets

Throughout this paper, let X and Y be two nonempty sets, $\mathfrak{F}(X)=\{\underset{\sim}{A};\ \underset{\sim}{A}: X\to[0,1]\}$ be the class of all fuzzy subsets of X, analogously, $\mathfrak{F}(Y)$ (resp. $\mathfrak{F}(X\times Y)$) be the class of all fuzzy subsets of Y (resp. $X\times Y$), and we make the convention: $0\cdot\infty=0$.

The following definitions and conclusions are introduced from[3,4,5,6,7].

Definition1.1 The nonempty subset $\underset{\sim}{\mathfrak{F}}$ of $\mathfrak{F}(X)$ is called a fuzzy σ-algebra, if the following conditions are satisfied:
(1) Φ, $X\in\underset{\sim}{\mathfrak{F}}$;

(2) If $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, then $\underset{\sim}{A}^c \in \underset{\sim}{\mathfrak{F}}$;

(3) If $\{\underset{\sim}{A}_n\} \subset \underset{\sim}{\mathfrak{F}}$, then $\bigcup_{n=1}^{\infty} \underset{\sim}{A}_n \in \underset{\sim}{\mathfrak{F}}$.

Denote $\underset{\sim}{M} = \{f;\ f: X \longrightarrow (-\infty, \infty),\ \{x; f(x) \geq \alpha\} \in \underset{\sim}{\mathfrak{F}}, \alpha \in (-\infty, \infty)\}$, i.e. $\underset{\sim}{M}$ is the set of all measurable functions on $\underset{\sim}{\mathfrak{F}}$, $\underset{\sim}{M}^+ = \{f; f \in \underset{\sim}{M}, f \geq 0\}$, $\mathfrak{B} = \{E;$ E is the classical set in $\underset{\sim}{\mathfrak{F}}\}$. Evidently, $\mathfrak{B}$ is a classical σ-algebra such that $\mathfrak{B} \subset \underset{\sim}{\mathfrak{F}}$, and all functions in $\underset{\sim}{M}$ are measurable on $\mathfrak{B}$.

Definition1.2 Let $\underset{\sim}{\mu}: \underset{\sim}{\mathfrak{F}} \longrightarrow [0, \infty]$ be a fuzzy measure on $\underset{\sim}{\mathfrak{F}}$ (cf.[4,7]), it is called σ-additive, if we have

$$\underset{\sim}{\mu}\left(\bigcup_{n=1}^{\infty} \underset{\sim}{A}_n\right) = \sum_{n=1}^{\infty} \underset{\sim}{\mu}(\underset{\sim}{A}_n) \quad \text{whenever } \{\underset{\sim}{A}_n\} \subset \underset{\sim}{\mathfrak{F}},\ \underset{\sim}{A}_i \cap \underset{\sim}{A}_j = \Phi,\ i \neq j.$$

In this paper, we shall always assume that $\underset{\sim}{\mu}$ is a σ-additive fuzzy measure.

Definition1.3 (1) Let $f = \sum_{i=1}^{n} \alpha_i E_i \in \underset{\sim}{M}^+$ be a simple function (cf.[3,4]), $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$. The abstract integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is defined by

$$\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \triangleq \sum_{i=1}^{n} \alpha_i \underset{\sim}{\mu}(\underset{\sim}{A} \cap E_i);$$

(2) If $f \in \underset{\sim}{M}^+$ is an arbitrary measurable function, $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, then the abstract integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is defined by $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \triangleq \sup\{\int_{\underset{\sim}{A}} s d\underset{\sim}{\mu};\ 0 \leq s \leq f$, s is the simple function$\}$;

(3) If $f \in \underset{\sim}{M}$, $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, and if $\int_{\underset{\sim}{A}} f^+ d\underset{\sim}{\mu} < \infty$ or $\int_{\underset{\sim}{A}} f^- d\underset{\sim}{\mu} < \infty$, then we say the abstract integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is existent, and the abstract integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is defined by $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \triangleq \int_{\underset{\sim}{A}} f^+ d\underset{\sim}{\mu} - \int_{\underset{\sim}{A}} f^- d\underset{\sim}{\mu}$, where $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$. If $\left|\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}\right| < \infty$,then we say f is integrable on $\underset{\sim}{A}$.

Such a abstract integral on fuzzy set holds all elementary properties of the classical integral given in [1,2], and we obtained some convergence theorems and transformation theorems for the abstract integrals on fuzzy sets. (cf.[3,4,5,6])

§2 Product Fuzzy Measurable Space

In the section, we shall prove the unique existence theorem of the fuzzy σ-algebra generated by a class of fuzzy sets, and we shall introduce the Cartesian product for fuzzy sets and the product fuzzy σ-algebra.

Theorem2.1 If $\mathcal{A}$ is any subclass of $\mathfrak{F}(X)$, then there exists a unique fuzzy σ-algebra $\sigma(\mathcal{A})$ such that

(1) $\mathcal{A} \subset \sigma(\mathcal{A})$;

(2) If $\underset{\sim}{\mathfrak{F}}$ is any other fuzzy σ-algebra containing $\mathcal{A}$, then $\sigma(\mathcal{A}) \subset \underset{\sim}{\mathfrak{F}}$.

The fuzzy σ-algebra $\sigma(\mathcal{A})$, the smallest fuzzy σ-algebra containing $\mathcal{A}$, is called the fuzzy σ-algebra generated by $\mathcal{A}$.

Now we give the concept of the Cartesian product of two fuzzy sets.

Definition2.2 Let $\underset{\sim}{A} \in \mathfrak{F}(X)$, $\underset{\sim}{B} \in \mathfrak{F}(Y)$, the Cartesian product of $\underset{\sim}{A}$ and $\underset{\sim}{B}$ is defined by

$$(\underset{\sim}{A} \times \underset{\sim}{B})(x,y) \triangleq (\text{supp } \underset{\sim}{A})(x) \wedge \underset{\sim}{B}(y) \quad \text{for any } (x,y) \in X \times Y.$$

Obviously, $\underset{\sim}{A} \times \underset{\sim}{B}$ is the fuzzy set in $\mathfrak{F}(X \times Y)$.

Remark: When $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are the classical sets, $\underset{\sim}{A} \times \underset{\sim}{B}$ defined in Definition2.2 coincides with the Cartesian product for the classical sets given in [1,2]. (cf. [1,2])

We shall assume that $\underset{\sim}{\mathfrak{F}}$ is a fuzzy σ-algebra on X, $\underset{\sim}{\mathfrak{H}}$ is a fuzzy σ-algebra on Y .

Definition2.3 Let $\mathcal{C} = \{\underset{\sim}{A} \times \underset{\sim}{B};\ \underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}},\ \underset{\sim}{B} \in \underset{\sim}{\mathfrak{H}}\}$ (the elements in $\mathcal{C}$ are called the measurable rectangles), the fuzzy σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$ (i.e. the smallest fuzzy σ-algebra containing $\mathcal{C}$) is called the product fuzzy σ-algebra of $\underset{\sim}{\mathfrak{F}}$ and $\underset{\sim}{\mathfrak{H}}$, and denote it by $\underset{\sim}{\mathfrak{F}} \times \underset{\sim}{\mathfrak{H}}$, the measurable space $(X \times Y, \underset{\sim}{\mathfrak{F}} \times \underset{\sim}{\mathfrak{H}})$ is called the product fuzzy measurable space of $(X, \underset{\sim}{\mathfrak{F}})$ and $(Y, \underset{\sim}{\mathfrak{H}})$.

§3 Sections of Fuzzy Sets and Sections of Functions

In the section, we shall introduce the sections of fuzzy sets and the sections of functions, some properties of the sections will be studied.

Definition3.1 Let $\underset{\sim}{D} \in \mathfrak{F}(X \times Y)$. For any given $x \in X$, the fuzzy set $\underset{\sim}{D}_x$: $\underset{\sim}{D}_x(y) \triangleq \underset{\sim}{D}(x,y)$ for any $y \in Y$, is called the section of $\underset{\sim}{D}$ at x. For any given $y \in Y$, the fuzzy set $\underset{\sim}{D}_y$: $\underset{\sim}{D}_y(x) \triangleq \underset{\sim}{D}(x,y)$ for any $x \in X$, is called the section of $\underset{\sim}{D}$ at y.

Obviously, $\underset{\sim}{D}_x \in \mathfrak{F}(Y)$, $\underset{\sim}{D}_y \in \mathfrak{F}(X)$.

Remark: When $\underset{\sim}{D}$ is the classical set, the $\underset{\sim}{D}_x$ and $\underset{\sim}{D}_y$ defined in Definition3.1 coincide with the sections for the classical sets given in [1,2]. (cf. [1,2])

Theorem3.2 The sections of fuzzy sets possess the following elementary properties:

(1) Let $\underset{\sim}{D}^1, \underset{\sim}{D}^2 \in \mathfrak{F}(X \times Y)$, $x \in X$, $y \in Y$, then

$$\underset{\sim}{D}^1 \cap \underset{\sim}{D}^2 = \Phi \Longrightarrow \underset{\sim}{D}^1_x \cap \underset{\sim}{D}^2_x = \Phi \ , \ \underset{\sim}{D}^1_y \cap \underset{\sim}{D}^2_y = \Phi \ ;$$

$$\underset{\sim}{D}^1 \subset \underset{\sim}{D}^2 \Longrightarrow \underset{\sim}{D}^1_x \subset \underset{\sim}{D}^2_x \ , \ \underset{\sim}{D}^1_y \subset \underset{\sim}{D}^2_y \ ;$$

(2) Let $\underset{\sim}{D} \in \mathfrak{F}(X \times Y)$, $x \in X$, $y \in Y$, then

$$(\underset{\sim}{D}^c)_x = (\underset{\sim}{D}_x)^c, \quad (\underset{\sim}{D}^c)_y = (\underset{\sim}{D}_y)^c$$

(3) Let $\{\underset{\sim}{D}^m\} \subset \mathfrak{F}(X \times Y)$, $x \in X$, $y \in Y$, then

$$(\bigcup_{m=1}^{\infty} \underset{\sim}{D}^m)_x = \bigcup_{m=1}^{\infty} \underset{\sim}{D}^m_x \ , \quad (\bigcup_{m=1}^{\infty} \underset{\sim}{D}^m)_y = \bigcup_{m=1}^{\infty} \underset{\sim}{D}^m_y \ ,$$

$$(\bigcap_{m=1}^{\infty} \underset{\sim}{D}^m)_x = \bigcap_{m=1}^{\infty} \underset{\sim}{D}^m_x \quad (\bigcap_{m=1}^{\infty} \underset{\sim}{D}^m)_y = \bigcap_{m=1}^{\infty} \underset{\sim}{D}^m_y \ .$$

Theorem3.3 Let $\underset{\sim}{A} \in \mathfrak{F}(X)$, $\underset{\sim}{B} \in \mathfrak{F}(Y)$, then

$$(\underset{\sim}{A} \times \underset{\sim}{B})_x = \begin{cases} \underset{\sim}{B} & \text{if } x \in \operatorname{supp} \underset{\sim}{A} \\ \Phi & \text{if } x \notin \operatorname{supp} \underset{\sim}{A} \end{cases}, \text{ and}$$

$(\underset{\sim}{A} \times \underset{\sim}{B})_y$ is the fuzzy set such that

$$(\underset{\sim}{A} \times \underset{\sim}{B})_y(x) = \begin{cases} a & \text{if } x \in \operatorname{supp} \underset{\sim}{A} \\ o & \text{if } x \notin \operatorname{supp} \underset{\sim}{A} \end{cases} \quad \text{for any } x \in X,$$

where $a = \underset{\sim}{B}(y)$ is a constant such that $0 \leqslant a \leqslant 1$.

Theorem3.4 Suppose that $\underset{\sim}{\mathfrak{F}}$ is the fuzzy σ-algebra containing all fuzzy sets of the form $\underset{\sim}{E}$:

$$\underset{\sim}{E}(x) = \begin{cases} a & \text{if } x \in \operatorname{supp} \underset{\sim}{A} \\ o & \text{if } x \notin \operatorname{supp} \underset{\sim}{A} \end{cases} \quad \text{for any } x \in X,$$

where $\underset{\sim}{A}$ is any fuzzy set in $\underset{\sim}{\mathfrak{F}}$, a is any constant such that $o < a \leqslant 1$. If $\underset{\sim}{D} \in \underset{\sim}{\mathfrak{F}} \times \underset{\sim}{\mathfrak{H}}$, then

(1) $\underset{\sim}{D}_x \in \underset{\sim}{\mathfrak{H}}$ for every $x \in X$;

(2) $\underset{\sim}{D}_y \in \underset{\sim}{\mathfrak{F}}$ for every $y \in Y$.

Proposition3.5 Let $\underset{\sim}{\mathfrak{F}}$ be a fuzzy σ-algebra on X, then the following statements are equivalent:

(1) $\underset{\sim}{\mathfrak{F}}$ contains all fuzzy sets of the form $\underset{\sim}{E}$:

$$\underset{\sim}{E}(x)=\begin{cases} a & \text{if } x \in \text{supp } \underset{\sim}{A} \\ o & \text{if } x \notin \text{supp } \underset{\sim}{A} \end{cases} \quad \text{for any } x \in X,$$

where $\underset{\sim}{A}$ is any fuzzy set in $\underset{\sim}{\mathfrak{F}}$, a is any constant such that $o \leqslant a \leqslant 1$.

(2) Whenever $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, $a \in [o,1]$, then supp $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, $a \in \underset{\sim}{\mathfrak{F}}$, where a is the fuzzy set: $a(x)\equiv a$ for any $x \in X$.

We turn now to the study of the sections of functions.

Definition3.6 Let $g: X\times Y \to (-\infty, \infty)$ be a function on $X\times Y$. For any given $x \in X$, the function g_x: $g_x(y) \triangleq g(x,y)$ for any $y \in Y$, is called the section of g at x. For any given $y \in Y$, the function g_y: $g_y(x) \triangleq g(x,y)$ for any $x \in X$, is called the section of g at y.

We have the following theorem for the sections of functions.

Theorem3.7 Let $\underset{\sim}{\mathfrak{F}}$ be the fuzzy σ-algebra given in Theorem 3.4, g be a measurable function on $\underset{\sim}{\mathfrak{F}}\times\underset{\sim}{\mathfrak{H}}$, then

(1) g_x is a measurable function on $\underset{\sim}{\mathfrak{H}}$ for every given $x \in X$;

(2) g_y is a measurable function on $\underset{\sim}{\mathfrak{F}}$ for every given $y \in Y$.

§4 Product Fuzzy Measure

Throughout this section, we shall assume that $\underset{\sim}{\mathfrak{H}}$ is a fuzzy σ-algebra on Y, $\underset{\sim}{\nu}$ is a totally finite (i.e. $\underset{\sim}{\nu}(Y)<\infty$) and σ-additive fuzzy measure on $\underset{\sim}{\mathfrak{H}}$, $\underset{\sim}{\mathfrak{F}}$ is the fuzzy σ-algebra such that

(1) $\underset{\sim}{\mathfrak{F}}$ satisfies the condition (1) given in Proposition3.5;

(2) $f(x)=\underset{\sim}{\nu}(\underset{\sim}{D}_x) \in \underset{\sim}{M}$ for every $\underset{\sim}{D} \in \underset{\sim}{\mathfrak{F}}\times\underset{\sim}{\mathfrak{H}}$.

And suppose that $\underset{\sim}{\mu}$ is a totally finite (i.e. $\underset{\sim}{\mu}(X)<\infty$) and

σ-additive fuzzy measure on $\underset{\sim}{\mathfrak{F}}$,

$\mathfrak{B} \triangleq \{E;$ E is the classical set in $\underset{\sim}{\mathfrak{F}}\}$,

$\mathfrak{D} \triangleq \{F;$ F is the classical set in $\underset{\sim}{\mathfrak{H}}\}$.

Evidently, all measurable functions on $\underset{\sim}{\mathfrak{F}}$ are measurable on $\mathfrak{B}$, all measurable functions on $\underset{\sim}{\mathfrak{H}}$ are measurable on $\mathfrak{D}$, and all measurable functions on $\mathfrak{B}\times\mathfrak{D}$ are measurable on $\underset{\sim}{\mathfrak{F}}\times\underset{\sim}{\mathfrak{H}}$.

By using the results given in [4,5,6], we can prove the following theorem.

Theorem4.1 Let $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, if we define

$$\underset{\sim}{\lambda}_{\underset{\sim}{A}}(\underset{\sim}{D}) \triangleq \int_{\underset{\sim}{A}} \underset{\sim}{\nu}(\underset{\sim}{D}_x)\,d\underset{\sim}{\mu} \quad \text{for every } \underset{\sim}{D} \in \underset{\sim}{\mathfrak{F}}\times\underset{\sim}{\mathfrak{H}},$$

then $\lambda_{\underset{\sim}{A}}$ is a totally finite and σ-additive fuzzy measure on $\underset{\sim}{\mathfrak{F}}\times\underset{\sim}{\mathfrak{H}}$.

Definition4.2 The $\underset{\sim}{\lambda}_{\underset{\sim}{A}}$ given in Theorem4.1 is called the product fuzzy measure of $\underset{\sim}{\mu}$ and $\underset{\sim}{\nu}$ with respect to $\underset{\sim}{A}$, denote it by $(\underset{\sim}{\mu}\times\underset{\sim}{\nu})_{\underset{\sim}{A}}$. $(X\times Y, \underset{\sim}{\mathfrak{F}}\times\underset{\sim}{\mathfrak{H}}, (\underset{\sim}{\mu}\times\underset{\sim}{\nu})_{\underset{\sim}{A}})$ is called the product fuzzy measure space of $(X,\underset{\sim}{\mathfrak{F}},\underset{\sim}{\mu})$ and $(Y,\underset{\sim}{\mathfrak{H}},\underset{\sim}{\nu})$ with respect to $\underset{\sim}{A}$.

In the following, we shall always assume that μ^*, ν^* and λ^* denote such classical measures respectively:

for any given $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$, $\underset{\sim}{B}\in\underset{\sim}{\mathfrak{H}}$,

$\mu^*(E) \triangleq \underset{\sim}{\mu}(\underset{\sim}{A} \cap E)$ for any $E \in \mathfrak{B}$,

$\nu^*(F) \triangleq \underset{\sim}{\nu}(\underset{\sim}{B} \cap F)$ for any $F \in \mathfrak{D}$,

$\lambda^*(D) \triangleq (\underset{\sim}{\mu}\times\underset{\sim}{\nu})_{\underset{\sim}{A}}[(\underset{\sim}{A}\times\underset{\sim}{B})\cap D]$ for any $D \in \mathfrak{B}\times\mathfrak{D}$.

We have the following theorem for the classical measures μ^*, ν^* and λ^*.

Theorem4.3 $\lambda^* = \mu^*\times\nu^*$,

where $(\mu^*\times\nu^*)(D) = \int_X \nu^*(D_x)\,d\mu^*$ for every $D\in\mathfrak{B}\times\mathfrak{D}$ is the product measure of μ^* and ν^* defined in [1,2]. (cf. [1,2])

§ 5 Fubini's Theorems of Abstract Integrals on Fuzzy Sets

In the section, we shall prove the Fubini's theorems of abstract integrals on fuzzy sets.

All signs used in this section coincide with those given in Section 4.

Theorem5.1 Let $g(x,y)$ be a nonnegative measurable function on $\mathfrak{B}\times\mathfrak{D}$, $\underset{\sim}{A}\in\mathfrak{F}$, $\underset{\sim}{B}\in\mathfrak{H}$, and

$$\varphi(y)=\int_{\underset{\sim}{A}}g(x,y)\,d\underset{\sim}{\mu}\overset{\Delta}{=}\int_{\underset{\sim}{A}}g_y(x)\,d\underset{\sim}{\mu}<\infty,$$

$$\psi(x)=\int_{\underset{\sim}{B}}g(x,y)\,d\underset{\sim}{\nu}\overset{\Delta}{=}\int_{\underset{\sim}{B}}g_x(y)\,d\underset{\sim}{\nu}<\infty,$$

then $\varphi(y)$ is a nonnegative measurable function on $\mathfrak{H}$, $\psi(x)$ is a nonnegative measurable function on $\mathfrak{F}$.

[1,2] gave the following result. (cf. [1,2])

Theorem5.2 (Fubini's Theorem) Let $g(x,y)$ be a nonnegative measurable function on $\mathfrak{B}\times\mathfrak{D}$, $\int_X g(x,y)\,du^*<\infty$, $\int_Y g(x,y)\,d\nu^*<\infty$,

then $\int_{X\times Y}g(x,y)\,d(u^*\times\nu^*)=\int_X(\int_Y g(x,y)\,d\nu^*)\,du^*=\int_Y(\int_X g(x,y)\,du^*)\,d\nu^*$.

We have the similar conclusions for the abstract integrals on fuzzy sets.

Theorem5.3 (Fubini's Theorem) Let $g(x,y)$ be a nonnegative measurable function on $\mathfrak{B}\times\mathfrak{D}$, $\underset{\sim}{A}\in\mathfrak{F}$, $\underset{\sim}{B}\in\mathfrak{H}$, and

$$\int_{\underset{\sim}{A}}g(x,y)\,d\underset{\sim}{\mu}<\infty,\quad \int_{\underset{\sim}{B}}g(x,y)\,d\underset{\sim}{\nu}<\infty, \text{ then}$$

$$\int_{\underset{\sim}{A}\times\underset{\sim}{B}}g(x,y)\,d(\underset{\sim}{\mu}\times\underset{\sim}{\nu})_{\underset{\sim}{A}}=\int_{\underset{\sim}{A}}(\int_{\underset{\sim}{B}}g(x,y)\,d\underset{\sim}{\nu})\,d\underset{\sim}{\mu}=\int_{\underset{\sim}{B}}(\int_{\underset{\sim}{A}}g(x,y)\,d\underset{\sim}{\mu})\,d\underset{\sim}{\nu}.$$

Theorem5.4 (Fubini's Theorem) Let $g(x,y)$ be an arbitrary measurable function on $\mathfrak{B}\times\mathfrak{D}$, $\underset{\sim}{A}\in\mathfrak{F}$, $\underset{\sim}{B}\in\mathfrak{H}$, and $g(x,y)$ is integrable on $\underset{\sim}{A}$, $\underset{\sim}{B}$ and $\underset{\sim}{A}\times\underset{\sim}{B}$, then

$$\int_{\underset{\sim}{A}\times\underset{\sim}{B}}g(x,y)\,d(\underset{\sim}{\mu}\times\underset{\sim}{\nu})_{\underset{\sim}{A}}=\int_{\underset{\sim}{A}}(\int_{\underset{\sim}{B}}g(x,y)\,d\underset{\sim}{\nu})\,d\underset{\sim}{\mu}=\int_{\underset{\sim}{B}}(\int_{\underset{\sim}{A}}g(x,y)\,d\underset{\sim}{\mu})\,d\underset{\sim}{\nu}.$$

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