SOME FIXED DEGREE THEOREMS OF GENERALIZED FUZZY MAPPING

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This paper obtains some new fixed degree theorems of fuzzy mapping by concept of fixed degree of generalized fuzzy mappings on the basis of [3].

1. Preliminaries

Throughout this paper (X,d) denotes a complete metric space; $H(\cdot,\cdot)$, the Hausdorff metric induced by metric d; CB(X), the collection of all non-empty bounded closed sets of X; $\mathcal{F}(X)$, the collection of all fuzzy sets in X. Let $A\in\mathcal{F}(X)$, $\alpha\in(0,1]$, we write

$$\mathrm{Supp}A=\{x\in X: A(x)>0\};$$

$$A_\alpha=\{x\in X: A(x)=\alpha\};$$

$$\langle A\rangle_\alpha=\{x\in X: A(x)\geqslant\alpha\};$$

$$\tilde{A}=\{\xi_\lambda^x: x\in X,\ A(x)=\lambda\in(0,1]\}.$$

where $\{\xi_\lambda^x\}\subset\mathcal{F}(X)$, ξ_λ^x is a fuzzy point which takes x as supporting point, λ as value, i.e.

$$\xi_\lambda^x(u)=\begin{cases}\lambda & u=x\\ 0 & u\neq x\end{cases}\qquad \text{for } u\in X.$$

Definition1.1. Let $A\in\mathcal{F}(X)$, the mapping $F: \tilde{A}\longrightarrow\mathcal{F}(X)$ is called fuzzy mapping over A. If for each $\xi_\lambda^x\in\tilde{A}$, we have $F(\xi_\lambda^x)\subset A$ For convenience' sake, we write $F(\xi_\lambda^x)=F_{\xi_\lambda^x}$, thus $F_{\xi_\lambda^x}(y)$ denotes the membership degree of the point $y\in X$ with respect to the fuzzy set $F_{\xi_\lambda^x}$.

Definition1.2. Let $A\in\mathcal{F}(X)$, F be a fuzzy mapping over A, $\xi_\lambda^x\in\tilde{A}$, if $F_{\xi_\lambda^x}(x)=\alpha$, then $\frac{\alpha}{\lambda}$ is called fixed degree of ξ_λ^x for fuzzy mapping F. we write

$$D_{fix}(\xi_\lambda^x,F)=\frac{\alpha}{\lambda}.$$

Specifically, if $D_{fix}(\xi_\lambda^x,F)=1$, i.e. $F_{\xi_\lambda^x}(x)=\lambda$, then ξ_λ^x is called fixed point of F. If $F_{\xi_\lambda^x}(x)=\max_{u\in X}F_{\xi_\lambda^x}(u)$, then we say that F obtains maximal fixed degree at fuzzy point ξ_λ^x.

Let $A\in\mathcal{F}(X)$, F be a fuzzy mapping over A. If for any $x\in$ suppA there exists a corresponding $\alpha(x)\in(0,1]$ such that $\{y\in X: F_{\xi^x_{A(x)}}(y)=\alpha(x)\}\in CB(X)$, we can define a set-valued mapping $\hat{F}$: $\mathrm{Supp}A\longrightarrow CB(X)$ as follows:

$$\hat{F}(x)=\{y\in X:\ F_{\xi^x_{A(x)}}(y)=\alpha(x)\}\qquad \text{for}\ \forall\, x\in \mathrm{Supp}A \quad (1.1)$$

Clearly, for any $x\in$ AuppA, we have $\hat{F}(x)\subset$ SuppA, thus for any $y\in\hat{F}(x)$ we have $\xi^y_{A(y)}\in\tilde{A}$. From the definition we can immediately obtain the following result.

Lemma1.1([3]). Let $A\in\mathcal{F}(X)$, F be a fuzzy mapping over A, $\hat{F}$ be the set-valued mapping defined by F according to (1.1). Then fixed degree of $\xi^x_{A(x)}\in\tilde{A}$ with respect to F is equal to $\frac{\alpha(x)}{A(x)}$ if and only if x is fixed point of the set-valued mapping $\hat{F}$, i.e. $x\in\hat{F}(x)=\{y\in X:\ F_{\xi^x_{A(x)}}(y)=\alpha(x)\}$.

Lemma1.2([4]). Let A, $B\in CB(X)$, then for any $a\in A$, certainly there exists a point $b\in B$ such that

(1). For any $\varepsilon>0$, we have $d(a,b)\le H(A,B)+\varepsilon$;

(2). For any $\gamma>1$, we have $d(a,b)\le\gamma H(A,B)$.

2. Main resulte

Theorem2.1. Let $A\in\mathcal{F}(X)$, $\langle A\rangle_\gamma\in CB(X)$, $0<\gamma<1$, F,G be two fuzzy mappings over A, if for any $x,y\in \mathrm{Supp}A$ there are corresponding $\alpha(x)$, $\beta(y)\in[\gamma,1]$ such that $(F_{\xi^x_{A(x)}})_{\alpha(x)}$, $(G_{\xi^y_{A(y)}})_{\beta(y)}$ $\in CB(X)$, and

$$H((F_{\xi^x_{A(x)}})_{\alpha(x)},(G_{\xi^y_{A(y)}})_{\beta(y)})\le ad(x,y)+b[d(x,(F_{\xi^x_{A(x)}})_{\alpha(x)})+d(y,(G_{\xi^y_{A(y)}})_{\beta(y)})]+c[d(x,(G_{\xi^y_{A(y)}})_{\beta(y)})+d(y,(F_{\xi^x_{A(x)}})_{\alpha(x)})].$$

$$\forall\ x,y\in\mathrm{Supp}A$$

where $a,b,c\geqslant0$, $a+2b+2c<1$ (a,b,c can be function of x,y too). Then there exists a fuzzy point $\xi^{x^*}_{A(x^*)}\in\widetilde{A}$ such that the common fixeddegree of $\xi^{x^*}_{A(x^*)}$ for F and G is equal to $\min\left\{\frac{\alpha(x^*)}{A(x^*)},\frac{\beta(x^*)}{A(x^*)}\right\}$, i.e.

$$D_{fix}(\xi^{x^*}_{A(x^*)},F\cap G)=\min\left\{\frac{\alpha(x^*)}{A(x^*)},\frac{\beta(x^*)}{A(x^*)}\right\}.$$

Proof. Let $\hat{F},\hat{G}$: $\mathrm{Supp}A\longrightarrow CB(X)$ be two set-valued mappings defined by F and G according to (1.1) respectively.By using lemma1.1, suffice it to prove that there exists $x^*\in\mathrm{Supp}A$ such that $x^*\in(\hat{F}(x^*)\cap\hat{G}(x^*))$.

In fact, for any given $x_0\in\mathrm{Supp}A$, by using definition(1.1) of $\hat{F}$, then $\hat{F}(x_0)\subset\mathrm{Supp}A$. Taking $x_1\in\hat{F}(x_0)$, then $x_1\in\langle A\rangle_\gamma\subset\mathrm{Supp}A$. In fact, $\xi^{x_0}_{A(x_0)}\in\widetilde{A}$, $F_{\xi^{x_0}_{A(x_0)}}\subset A$, follows from definitions of F and $\hat{F}$ that

$$r \leq \alpha(x_0)=F_{\substack{x_0 \\ \xi A(x_0)}}(x_1) \leq A(x_1).$$

Let $\alpha > H(\hat{F}(x_0),\hat{G}(x_1))$ by using lemma1.2(1) there exists $x_2 \in \hat{G}(x_1)$ $(x_2 \in \langle A\rangle_r)$ such that $d(x_1,x_2) \leq \alpha$,and

$H(\hat{F}(x_2),\hat{G}(x_1)) \leq ad(x_1,x_2)+b[d(x_2,\hat{F}(x_2))+d(x_1,\hat{G}(x_1))]+c[d(x_2,\hat{G}(x_1))+d(x_1,\hat{F}(x_2))$

$\leq ad(x_1,x_2)+b[H(\hat{G}(x_1),\hat{F}(x_2))+d(x_1,x_2)+0]+c[0+d(x_1,x_2)+H(\hat{G}(x_1),\hat{F}(x_2))$

$$\leq \frac{a+b+c}{1-b-c}\, d(x_1,x_2) \leq \beta\alpha$$

where let $\frac{a+b+c}{1-b-c}=\beta$, since $a+2b+2c<1$, $a,b,c \geq 0$, hence, $0<\frac{a+b+c}{1-b-c}=\beta<1$.

By using lemma1.2(1), there exists $x_3 \in \hat{F}(x_2)$ $(x_3 \in \langle A\rangle_r)$such that $d(x_2,x_3) \leq \beta\alpha$. and

$H(\hat{F}(x_2),\hat{G}(x_3)) \leq ad(x_2,x_3)+b[d(x_2,\hat{F}(x_2))+d(x_3,\hat{G}(x_3))]+c[d(x_2\ \hat{G}(x_3))+d(x_3,\hat{F}(x_2))]$

$\leq ad(x_2,x_3)+b[d(x_2,x_3)+H(\hat{F}(x_2),\hat{G}(x_3))]+c[d(x_2\ x_3)+H(\hat{F}(x_2),\hat{G}(x_3))]$

$$\leq \frac{a+b+c}{1-b-c}\, d(x_2,x_3) \leq \beta^2\alpha$$

Continuing in this way we can produce a sequence $\{x_n\}_{n=1}^{\infty} \subset \langle A\rangle_r$ such that

$$x_{2n} \in \hat{G}(x_{2n-1}),\ n=1,2,\cdots$$
$$x_{2n+1} \in \hat{F}(x_{2n}),\ n=0,1,2,\cdots$$
$$d(x_n,x_{n+1}) \leq \beta^n\alpha \qquad \text{for } r \leq \alpha < 1,\ 0<\beta<1.$$

Next we prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. In fact for any positive integers k,m we have

$$d(x_{k+m},x_k)\leq \sum_{i=k}^{k+m-1} d(x_{i+1},x_i)\leq \sum_{i=k}^{k+m-1}\beta^i\alpha=\frac{\beta^k}{1-\beta}\alpha(0<\beta<1)$$

hence $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Let $x_n\longrightarrow x^*\in X$, since $\langle A\rangle_r\in CB(X)$, hence $x^*\in\langle A\rangle_r$, thus $x^*\in$ SuppA.

Now we prove that $x^*\in\hat{F}(x^*)$, i.e. $d(x^*,\hat{F}(x^*))=0$. In fact, if $d(x^*,\hat{F}(x^*))>0$ we have

$$d(x^*,\hat{F}(x^*))\leq d(x^*,x_{2n})+d(x_{2n},\hat{F}(x^*))$$
$$\leq d(x^*,x_{2n})+H(\hat{G}(x_{2n-1}),\hat{F}(x^*))$$
$$\leq d(x^*,x_{2n})+ad(x_{2n-1},x^*)+b[d(x_{2n-1},\hat{G}(x_{2n-1})+d(x^*,\hat{F}(x^*))]+c[d(x_{2n-1},\hat{F}(x^*))+d(x^*,\hat{G}(x_{2n-1}))]$$
$$\leq d(x^*,x_{2n})+ad(x_{2n-1},x^*)+b[d(x_{2n-1},x_{2n})+0+d(x^*,\hat{F}(x^*))+c[d(x_{2n-1},x^*)+d(x^*,F(x^*))+d(x^*,x_{2n})+0].$$

Letting $n\longrightarrow\infty$ on the right of inequality above, we have

$$d(x^*,\hat{F}(x^*))\leq(b+c)d(x^*,\hat{F}(x^*))$$

since $b+c<1$, this is a contradiction, it follows that $d(x^*\hat{F}(x^*))=0$ thus $x^*\in\hat{F}(x^*)$, i.e. $F_{\xi^{x^*}_{A(x^*)}}(x^*)=\alpha(x^*)$. thus

$$D_{fix}(\xi^{x^*}_{A(x^*)},F)=\frac{\alpha(x^*)}{A(x^*)};$$

similarly we can prove that

$$D_{fix}(\xi^{x^*}_{A(x^*)},G)=\frac{\beta(x^*)}{A(x^*)}$$

Therefore

$$D_{fix}(\xi^{x^*}_{A(x^*)},F\cap G)=\min\left\{\frac{\alpha(x^*)}{A(x^*)},\frac{\beta(x^*)}{A(x^*)}\right\}$$

Theorem2.2. Let $A\in\mathcal{F}(X)$, $\langle A\rangle_r\in CB(X)$, $0<r<1$, $\{F^i\}_{i=1}^{\infty}$ be a sequence of fuzzy mappings over A. If for any $x,y\in$ SuppA there are corrsponding $\alpha_i(x)\in[r,1]$ $(i=1,2,\cdots)$,

$$(F^i_{\xi^x_{A(x)}})_{\alpha_i(x)}\in CB(X),$$

and for any positive integers i,j, i≠j, we have

$$H((F^i\xi^x_{A(x)})_{\alpha_i(x)},(F^j\xi^y_{A(y)})_{\beta_j(y)})\le ad(x,y)+b[d(x,(F^i\xi^x_{A(x)})_{\alpha_i(x)})+d(y,(F^j\xi^y_{A(y)})_{\beta_j(y)})]+c[d(x,(F^j\xi^y_{A(y)})_{\beta_j(y)})+d(y,(F^i\xi^x_{A(x)})_{\alpha_i(x)})]$$
$$\forall\ x,y\ \mathrm{Supp}A$$

where a,b,c⩾0, a+2b+2c<1 (a,b,c can be functions of x,y too) Then there exists $\xi^{x^*}_{A(x^*)}\in\tilde{A}$ such that the common fixed degree of $\xi^{x^*}_{A(x^*)}$ for $\{F^i\}_{i=1}^{\infty}$ is equal to $\inf\{\frac{\alpha_i(x^*)}{A(x^*)}\}$.

The way pyoof of this theorem is analogus to the way of proof of theorem2.1.

Theorem2.3. Let $A\in\mathcal{H}(X)$, $\langle A\rangle_r\in CB(X)$, $0<r<1$, $\{F^i\}_{i=1}^{\infty}$ be a sequence of fuzzy mappings over A. For any x,y ∈ SuppA and any prositive integers i,j,i≠j, there are corresponding $\alpha_i(x)\in[r,1]$. we have $(F^i\xi^x_{A(x)})_{\alpha_i(x)}\in CB(X)$ and for any q∈(0,1),we have

$$H((F^i\xi^x_{A(x)})_{\alpha_i(x)},(F^j\xi^y_{A(y)})_{\beta_j(y)})\le q\max\{d(x,y),\ d(x,(F^i\xi^x_{A(x)})_{\alpha_i(x)}),\ d(y,(F^j\xi^y_{A(y)})_{\beta_j(y)}),\tfrac{1}{2}[d(x,(F^j\xi^y_{A(y)})_{\beta_j(y)})+d(y,(F^i\xi^x_{A(x)})_{\alpha_i(x)})]\}$$

Then there exists $\xi^{x^*}_{A(x^*)}\in\tilde{A}$ such that the common fixed degree of $\xi^{x^*}_{A(x^*)}$ for $\{F^i\}_{i=1}^{\infty}$ is equal to $\inf\{\frac{\alpha_i(x^*)}{A(x^*)}\}$

Proof. Let $\{\hat{F}_i\}_{i=1}^{\infty}$ be a sequence of set-valued mapping defined by $\{F^i\}_{i=1}^{\infty}$ according to (1.1) respectively. for any given $x_0\in$SuppA, we taking $x_1\in F_1(x_0)\subset$ SuppA. By using lemma1.2(1), Then $x_1\in\langle A\rangle_r\subset$ SuppA. By using lemma1.2(2),for any $\gamma:1<\gamma<\frac{1}{q}$ there exists $x_2\in\hat{F}_2(x_1)$ and $x_2\in\langle A\rangle_r\subset$ SuppA such that

$$d(x_1,x_2)\le\gamma H(F_1(x_0),F_2(x_1)).$$

By using lemma1.2(1),(2) again, there exists $x_3 \in \hat{F}_3(x_2)$ and $x_3 \in \langle A \rangle_r \subset \mathrm{Supp}A$, such that

$$d(x_2,x_3) \leq \Gamma H(\hat{F}_2(x_1),\hat{F}_3(x_2))$$

Continuing in this procedure we can produce a sequence $\{x_n\}_{n=1}^{\infty}$ SuppA and $x_n \in \langle A \rangle_r$ (n=1,2,$\cdots$) such that

$$x_n \in \hat{F}_n(x_{n-1}),\ n=1,2,\cdots$$
$$d(x_n,x_{n+1}) \leq \Gamma H(\hat{F}_n(x_{n-1}),\hat{F}_{n+1}(x_n)).$$

Next we prove that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.In fact, for any prositive integer n we have

$$d(x_n,x_{n+1}) \leq \Gamma H(\hat{F}_n(x_{n-1}),\hat{F}_{n+1}(x_n))$$
$$\leq \Gamma q\max\{d(x_{n-1},x_n),\ d(x_{n-1},\hat{F}_n(x_{n-1})),\ d(x_n,\hat{F}_{n+1}(x_n)),\ \tfrac{1}{2}[d(x_{n-1},\hat{F}_{n+1}(x_n))+d(x_n,\hat{F}_n(x_{n-1}))]\}$$
$$\leq \Gamma q\max\{d(x_{n-1},x_n),\ d(x_{n-1},x_n)+0,\ d(x_n,x_{n+1})+0,\ \tfrac{1}{2}[d(x_{n-1},x_n)+0+o]\}$$
$$\leq \Gamma q\max\{d(x_{n-1},x_n),\ d(x_n,x_{n+1})\}$$

since $\Gamma q<1$ hence for all prositive integer n we have

$$d(x_n,x_{n+1}) \leq \Gamma q d(x_{n-1},x_n) \leq \cdots \leq (\Gamma q)^n d(x_o,x_1)$$

For any prositive integers m,k, we have

$$d(x_{k+m},x_k) \leq \sum_{i=k}^{k+m-1} d(x_{i+1},x_i) \leq \sum_{i=k}^{k+m-1} (\Gamma q)^i d(x_o,x_1) \leq \frac{(\Gamma q)^k}{1-q} d(x_o,x_1)$$

Hence $\{F^i\}_{i=1}^{\infty}$ be a Cauchy sequence in X. Since $\langle A \rangle_r \in CB(X)$ hence there exists $x^* \in \langle A \rangle_r \subset \mathrm{Supp}A$ such that $x_n \longrightarrow x^*$.

Now we prove for any prositive integers i,n,i$\neq$n, $x^* \in \hat{F}_i(x^*)$ i.e. $d(x^*,\hat{F}_i(x^*))=0$.

Therwise, if $d(x^*,\hat{F}_i(x^*))>0$, we have

$$d(x^*,\hat{F}_i(x^*)) \leq d(x^*,x_n)+d(x_n,\hat{F}_i(x^*))$$
$$\leq d(x^*,x_n)+H(\hat{F}_n(x_{n-1}),\hat{F}_i(x^*))$$
$$\leq d(x^*,x_n)+q\max\{d(x_{n-1},x^*),\ d(x_{n-1},\hat{F}_n(x_{n-1}))$$
$$d(x^*,\hat{F}_i(x^*)),\ \tfrac{1}{2}[d(x_{n-1},\hat{F}_i(x^*))+d(x^*,\hat{F}_n(x_{n-1}))]\}$$
$$\leq d(x^*,x_n)+q\max\{d(x_{n-1},x^*),\ d(x_{n-1},x_n)+0,$$
$$d(x^*,\hat{F}_i(x^*)),\tfrac{1}{2}[d(x_{n-1},x^*)+d(x^*,\hat{F}_i(x^*))+d(x^*,x_n)+0]\}$$

Letting $n \longrightarrow \infty$ on the right side of inequality above, we have

$$d(x^*,\hat{F}_i(x^*)) \leq q d(x^*,\hat{F}_i(x^*))$$

since $q<1$ this is a contradiction. It follows that $d(x^*,\hat{F}_i(x^*))=0$. thus $x^* \in \hat{F}_i(x^*))(i=1,2,\cdots)$. By using lemma1.1 common fixed degree of $\{F^i\}_{i=1}^{\infty}$ is equal to $\inf\{\frac{\alpha_i(x^*)}{A(x^*)}\}$.

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