L-FUZZY IDEALS WITH A PRIMARY L-FUZZY DECOMPOSITION

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In this paper the concepts of a primary L-fuzzy decomposition of L-fuzzy ideals, an irredundant L-fuzzy decomposition and a normal L-fuzzy decomposition are introduced and some fundamental propositions proved. In particular, the uniqueness theorem for normal L-fuzzy decompositions of L-fuzzy ideals is given.

Keywords: Prime L-fuzzy ideal, primary L-fuzzy ideal, primary L-fuzzy decomposition, irredundant L-fuzzy decomposition, normal L-fuzzy decomposition.

1. INTRODUCTION

In [1, 2], we introduce some basic concepts of fuzzy ideal theory, as maximal L-fuzzy ideals, prime L-fuzzy ideals and primary L-fuzzy ideals and P-primary L-fuzzy ideals. These concepts are important to deal with decomposition problems of L-fuzzy ideals.

In this paper we discuss L-fuzzy ideals with a primary L-fuzzy decomposition. We define the concepts of primary L-fuzzy decomposition, irredundant L-fuzzy decomposition and normal L-fuzzy decomposition of L-fuzzy ideals which enable us to represent the theorem for L-fuzzy decompositions of L-fuzzy ideals. We prove that a decomposable L-fuzzy ideal will have only a finite number of minimal prime ideals,and that these will be associated with every primary L-fuzzy decomposition (Theorem 4.1). In particular, we show that if A is a decomposable L-fuzzy ideal then the base sets of the prime L-fuzzy ideals, which are associated with a normal L-fuzzy decomposition of A, depend only on X_A and not on the particular normal L-fuzzy decomposition considered.

2.PRIME L-FUZZY IDEALS AND PRIMARY L-FUZZY IDEALS

Throughout this paper $L=(L,\leqslant,\wedge,\vee)$ will be a completely distributive lattice with the least element $\emptyset$ and the greatest element 1. Let X be a nonempty (usual) set. An L-fuzzy set in X is a map $A: X\to L$, and the set of all L-fuzzy sets in X is denoted by

$$F(X)=\{A\,|\,A:X\to L\}$$

It is easily seen that $F(X)=(F(X),\leq,\wedge,\vee)$ is a completely distributive lattice with a least element $\mathbf{0}$ and a greatest element $\mathbf{1}$ in natural manner, where $\mathbf{0}(x)=\emptyset$, $\mathbf{1}(x)=1$ for all $x\in X$.

We recall some basic definitions and propositions in the paper [1,2] for reference purposes.

DEFINITION 2.1 Let $X=(X,+,\cdot)$ be a ring, and $A\in F(X)$, $A\neq\mathbf{0}$. A will be called an L-fuzzy ideal in X, iff

(1) $A(x)\wedge A(y)\leqslant A(x-y)$ for all $x,y\in X$;

(2) $A(x)\vee A(y)\leqslant A(x\cdot y)$ for all $x,y\in X$.

And I(X) will denote the set of all L-fuzzy ideals in X.

DEFINITION 2.2 Let A be an L-fuzzy subring of a ring X. Then the set

$$X_A=\{x\in X \mid A(x)=A(0)\}$$

where 0 is the zero element in X, is called a base set of A.

We will follow the convention that X is tacitly assumed to be a commutative ring with a unit element e and a zero element 0.

DEFINITION 2.3 Let $A\in I(X)$. A is called a prime L-fuzzy ideal, if for $x,y\in X$, $A(xy)=A(0)$ implies $A(x)=A(0)$ or $A(y)=A(0)$.

DEFINITION 2.4 Let $A\in I(X)$. A is called a primary L-fuzzy ideal in X, if for $x,y\in X$, $A(xy)=A(0)$ and $A(x)\neq A(0)$ implies $A(y^n)=A(0)$ for some positive integer n.

THEOREM 2.1 Let $P\in I(X)$ and Q be a given primary L-fuzzy ideal, and let $X_P=\{x \mid P(x)=P(0)\}$ denote the base set of P such that $Q(x^n)=Q(0)$ for at least one positive integal of n. Then

(1) P is a prime L-fuzzy ideal.

(2) $X_Q \subseteq X_P$.

(3) X_P is contained in base set of every other prime L-fuzzy ideal which contains X_Q.

(See [2, Theorem 3.1])

DEFINITION 2.5 If Q is a primary L-fuzzy ideal and P is the prime L-fuzzy ideal of Theorem 2.1,we shall say that Q belongs to P, and also that Q is a P-primary L-fuzzy ideal.Expressed in another way, if Q is a primary L-fuzzy ideal and P is the prime L-fuzzy ideal of Theorem 2.1, we shall say that X_P is a minimal prime ideal of the primary ideal X_Q.

THEOREM 2.2 Suppose that $P',Q'\in I(X)$ for which the following conditions are satisfied:

(1) $X_{Q'} \subseteq X_{P'}$.

(2) If $P'(x)=P'(0)$, then $Q'(x^n)=Q'(0)$ for some positive integer n.

(3) If $Q'(xy)=Q'(0)$ and $Q'(x)\neq Q'(0)$, then $P'(y)=P'(0)$.

In these circumstances P' is a prime L-fuzzy ideal, and Q' is a P'-primary L-fuzzy ideal.

(See [2, Theorem 3.2])

PROPOSITION 2.1 Q is a P-primary L-fuzzy ideal iff X_Q is a X_P-primary ideal.

(See [2, Proposition 3.3])

PROPOSITION 2.2 Let A, $B\in I(X)$. If Q is P-primary L-fuzzy ideal, $X_A X_B \subseteq X_Q$ and $X_A \nsubseteq X_Q$, then $X_B \subseteq X_P$.

(See [2, Proposition 3.2])

PROPOSITION 2.3 Let $A\in I(X)$. If Q is P-primary L-fuzzy ideal and $X_A \nsubseteq X_P$, then $X_Q:X_A=X_Q$.

Proof. By Proposition 2.1, X_Q is X_P-primary ideal. By [5, (4) of Proposition 1, p.7], $X_A(X_Q:X_A)\subseteq X_Q$, hence, by our hypotheses and by Proposition 2.2, $X_Q:X_A\subseteq X_Q$. Since, in any case, $X_Q\subseteq X_Q:X_A$, this proves our assertion.

PROPOSITION 2.4 If Q_1, Q_2, ..., Q_n are all of them P-primary L-fuzzy ideals, then $X_Q=X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_n}$ is a X_P-primary ideal.

Proof. By Proposition 2.1, X_{Q_1}, X_{Q_2}, ..., X_{Q_n} are all of them

X_P-primary ideals. Proposition 2.4 now follows from [4, Proposition 4, p.12].

PROPOSITION 2.5 Let $A \in I(X)$. If Q is a P-primary L-fuzzy ideal and if X_A is an ideal not contained in X_Q, then $X_Q:X_A$ is a X_P-primary ideal. If $X_A \subseteq X_Q$ then $X_Q:X_A=(e)$.

Proof. By [2, Proposition 2.2] and Proposition 2.1, X_A is an ideal and X_Q is X_P-primary ideal. If $X_A \subseteq X_Q$ then every element of the ring X is in $X_Q:X_A$, so that $X_Q:X_A=(e)$. Suppose now that $X_A \nsubseteq X_Q$ and put $X_{Q'}=X_Q:X_A$, we shall apply Theorem 2.2 to Q' and P. Since $X_A \nsubseteq X_Q$ we can find $a_o \in X_A$ so that $a_o \notin X_Q$. If now $y \in X_{Q'}$ then $a_o y \in X_Q$ and $a_o \notin X_Q$. By Proposition 2.1, X_Q is a X_P-primary ideal. Consequently $y \in X_P$, which proves that $X_{Q'} \subseteq X_P$. Again, if $P(x)=P(0)$, then with a suitable integer m we have $Q(x^m)=Q(0)$, i.e. $x^m \in X_Q \subseteq X_{Q'}$. Hence $Q'(x^m)=Q'(0)$. Finally, assume that $Q'(bc)=Q'(0)$ and that $Q(b) \neq Q(0)$, i.e. $bc \in X_{Q'}$ and $b \notin X_Q \subseteq X_{Q'}$. Then for any $a \in X_A$ we have $abc \in X_Q \subseteq X_{Q'}$ and $b \notin X_Q \subseteq X_{Q'}$, i.e. $Q'(abc)=Q'(0)$ and $Q'(b) \neq Q'(0)$, so that $ac \in X_P$, i.e. $P(ac)=P(0)$. By Theorem 2.2, Q' is P-primary L-fuzzy ideal. Therefore, $X_{Q'}=X_Q:X_A$ is a X_P-primary ideal from Proposition 2.1.

PROPOSITION 2.6 Let $A \in I(X)$ and P_1, P_2, ..., P_n be prime L-fuzzy ideals. If none of X_{P_1}, X_{P_2}, ..., X_{P_n} contains X_A, then there exists an element $a \in X_A$ such that no X_{P_i} contains a.

Proof. By [2, Proposition 2.2 and Theorem 2.2], X_A is an ideal and X_{P_1}, X_{P_2}, ..., X_{P_n} are prime ideals, hence, Proposition 2.6 follows from our hypotheses and by [5, Proposition 6, p.12].

3. THE L-FUZZY IDEAL GENERATED BY AN L-FUZZY SET

DEFINITION 3.1 Let $A\in F(X)$, $A\neq \mathbf{0}$. Then L-fuzzy set (A) is defined as follows:

$$(A)=\bigwedge_{A\leq B\in I(X)} B$$

It is clear that $(A)\in I(X)$, and (A) is called an L-fuzzy ideal generated by A.

In fact, we have, for any $x,y\in X$,

$$(A)(x)\wedge(A)(y)=\{\bigwedge_{A\leq B\in I(X)} B(x)\}\wedge\{\bigwedge_{A\leq C\in I(X)} C(y)\}$$

$$=\bigwedge_{\substack{A\leq B\wedge C\\ B,C\in I(X)}} B(x)\wedge C(y)$$

$$\leq\bigwedge_{A\leq B\in I(X)} B(x)\wedge B(y)$$

$$\leq\bigwedge_{A\leq B\in I(X)} B(x-y)$$

$$=(A)(x-y)$$

and

$$(A)(x)\vee(A)(y)=\{\bigwedge_{A\leq B\in I(X)} B(x)\}\vee\{\bigwedge_{A\leq C\in I(X)} C(y)\}$$

$$=\bigwedge_{\substack{A\leq B\wedge C\\ B,C\in I(X)}} B(x)\vee C(y)$$

$$\leq\bigwedge_{A\leq B\in I(X)} B(x)\vee B(y)$$

$$\leq\bigwedge_{A\leq B\in I(X)} B(xy)$$

$$=(A)(xy)$$

By Definition 2.1, $(A)\in I(X)$.

PROPOSITION 3.1 The L-fuzzy ideal $(\mathbf{1})$ is a prime L-fuzzy ideal, and $X_{(\mathbf{1})}=(e)$ and there are no $X_{(\mathbf{1})}$-primary ideals other than $X_{(\mathbf{1})}$

itself.

Proof. Assume that $(\mathbf{1})(xy)=(\mathbf{1})(0)$ for $x, y\in X$. On one hand,

$$1=\mathbf{1}(x)=\bigvee_{\mathbf{1}\leq B\in I(X)} B(x)=(\mathbf{1})(x)\leq(\mathbf{1})(x)\vee(\mathbf{1})(y)\leq(\mathbf{1})(xy);$$

on the other hand,

$$(\mathbf{1})(xy)=(\mathbf{1})(0)=\bigvee_{\mathbf{1}\leq B\in I(X)} B(0)=\mathbf{1}(0)=1$$

Thus in either case we have shown that $(\mathbf{1})(x)=(\mathbf{1})(0)$. By Definition 2.3, $(\mathbf{1})$ is a prime L-fuzzy ideal.

The second assertion of Proposition 3.1 is obvious.

4. DECOMPOSABLE L-FUZZY IDEALS

DEFINITION 4.1 Let $Q_1, Q_2, \ldots, Q_n$ be primary L-fuzzy ideals and the $A\in I(X)$. If X_A can be expressed in the form

$$X_A=X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_n},$$

we shall say that we have a primary L-fuzzy decomposition of A, and the individual Q_i will be called the primary L-fuzzy components of the decompsition.

THEOREM 4.1 Let $A\in I(X)$ and Q_i be P_i-primary L-fuzzy ideal for $1\leq i\leq n$. If $X_A=X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_n}$. Then any prime ideal which contains X_A must contain at least one of the X_{P_i}; the minimal prime ideals of X_A are just those prime ideals X_{P_i} which do not strictly contain any other X_{P_j}.

Proof. By Proposition 2.1, X_{Q_i} is X_{P_i}-primary ideal for $1\leq i\leq n$. Let X_P be a prime ideal containing X_A, then

$$X_{Q_1}X_{Q_2}\cdots X_{Q_n}\subseteq X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_n}\subseteq X_P,$$

consequently, by [2, Proposition 2.4], we can choose i so that $X_{Q_i} \subseteq X_P$, and then, by Theorem 2.1, $X_{P_i} \subseteq X_P$. This proves the first assertion, and the second assertion follows from the first simply by applying Definition 2.5.

Theorem 4.1 shows that a decomposable L-fuzzy ideal will have only a finite number of minimal prime ideals, and that these will be associated with every primary L-fuzzy decomposition. We shall now consider what further properties two different L-fuzzy decompositions of a given L-fuzzy ideal will have in common. Suppose that Q_i is P -primary L-fuzzy ideal for $1 \leq i \leq n$ and that

$$X_A = X_{Q_1} \cap X_{Q_2} \cap \cdots \cap X_{Q_n} ;$$

it may happen that X_{P_1}, X_{P_2}, ..., X_{P_n} are not all distinct. Let us suppose that

$$X_{P_{i_1}} = X_{P_{i_2}} = \cdots = X_{P_{i_r}} = X_P ,$$

then, by Proposition 2.4,

$$X_{Q_{i_1}} \cap X_{Q_{i_2}} \cap \cdots \cap X_{Q_{i_r}} = X_Q$$

is a P-primary ideal, so that we may replace all of $X_{Q_{i_1}}$, $X_{Q_{i_2}}, \cdots$, $X_{Q_{i_r}}$ by the single primary ideal X_Q . Again, if X_{Q_i} contains the intersection of the remaining X_{Q_j} it may be left out altogether.

DEFINITION 4.2 An L-fuzzy decomposition in which no base set X_{Q_i} of P_i-primary L-fuzzy ideal Q_i contains the intersection of the remaining base set X_{Q_j} of P_j-primary L-fuzzy ideal Q_j is called an irredundant L-fuzzy decomposition, and an irredandant L-fuzzy decomposition, in which the base sets of the prime L-fuzzy ideals belonging to the various primary L-fuzzy components are all different, is called a normal L-fuzzy decomposition.

THEOREM 4.2 Suppose that the L-fuzzy ideal A has a primary L-fuzzy decomposition, and let

$$X_A=X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_m}=X_{Q'_1}\cap X_{Q'_2}\cap\cdots\cap X_{Q'_n}$$

be two normal L-fuzzy decompositions of A, where Q_i is P_i-primary L-fuzzy ideal and Q'_j is P'_j-primary L-fuzzy ideal. Then m=n, and it is possible to number the L-fuzzy components in such a way that

$$X_{P_i}=X_{P'_i}$$

for $1\leq i\leq m=n$.

Proof. By Proposition 2.1, X_{Q_i} is X_{P_i}-primary ideal for $1\leq i\leq m$ and $X_{Q'_j}$ is $X_{P'_j}$-primary ideal for $1\leq j\leq n$. If A=(*1*) the assertion is trivial from Proposition 3.1. We may suppose therefore that A≠(*1*), in which case all the prime ideals

$$X_{P_1},\ X_{P_2},\cdots,\ X_{P_m},\ X_{P'_1},\ X_{P'_2},\cdots,\ X_{P'_n}$$

are proper. From this set of prime ideals we select one which is not strictly contained by any of the others. Without loss of generality we may suppose that the one which has been selected is X_{P_m}. We assert that X_{P_m} occurs among $X_{P'_1},\ X_{P'_2},\cdots,\ X_{P'_n}$. To prove this it will be enough to show that $X_{P_m}\subseteq X_{P'_j}$ for some j, and, by Theorem 2.1, the latter assertion will follow if we show that $X_{Q_m}\subseteq X_{P'_j}$ for some j. We shall now assume that for every j we have $X_{Q_m}\nsubseteq X_{P'_j}$ and derive a contradiction. By Proposition 2.3, $X_{Q'_j}:X_{Q_m}=X_{Q'_j}$, consequently, by [5, (5) of Proposition 1, p.7],

$$\begin{aligned}X_A:X_{Q_m}&=(X_{Q'_1}:X_{Q_m})\cap(X_{Q'_2}:X_{Q_m})\cap\cdots\cap(X_{Q'_n}:X_{Q_m})\\&=X_{Q'_1}\cap X_{Q'_2}\cap\cdots\cap X_{Q'_n}\\&=X_A\end{aligned}$$

But, on one hand, if $1\leq i<m$, $X_{P_m}\nsubseteq X_{P_i}$ (otherwise we should have $X_{P_m}=X_{P_i}$) and therefore $X_{Q_m}\nsubseteq X_{P_j}$, again by Proposition 2.3,

$X_{Q_i}:X_{Q_m}=X_{Q_i}$; and, on the other hand, $X_{Q_m}:X_{Q_m}=X_{(1)}=(e)$. These relation show that

$$X_A:X_{Q_m}=(X_{Q_1}:X_{Q_m})\cap(X_{Q_2}:X_{Q_m})\cap\cdots\cap(X_{Q_{m-1}}:X_{Q_m})\cap(X_{Q_m}:X_{Q_m})$$
$$=X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_{m-1}};$$

but, since we already know that $X_A:X_{Q_m}=X_A$, we have proved that

$$X_A=X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_{m-1}}.$$

This, however, contradicts the hypotheses that the given decompositions are normal L-fuzzy decompositioms.

Now that it has been proved the X_{P_m} occurs among $X_{P'_1}, X_{P'_2}, \cdots, X_{P'_n}$, we may, without loss of generality, suppose that $X_{P_m}=X_{P'_n}$. Put $X_Q=X_{Q_m}\cap X_{Q'_n}$ then, by Proposition 2.4, X_Q is a primary ideal belonging to $X_{P_m}=X_{P'_n}$. Also, $X_{Q_i}:X_Q=X_{Q_i}$ for $1\le i<m$ and $X_{Q_m}:X_Q=X_{(1)}=(e)$ --- the first relation follows from the fact that since $X_{P_m}\nsubseteq X_{P_i}$, X_Q is not contained in X_{P_i} , while the second follows from $X_Q\subseteq X_{Q_m}$ --- consequently

$$X_A:X_Q=X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_{m-1}};$$

An exactly similar argument shows that

$$X_A:X_Q=X_{Q'_1}\cap X_{Q'_2}\cap\cdots\cap X_{Q'_{n-1}};$$

hence

$$X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_{m-1}}=X_{Q'_1}\cap X_{Q'_2}\cap\cdots\cap X_{Q'_{m-1}}$$

and, moreover, both decompositions are normal L-fuzzy decompositions. We have, therefore, a situation entirely similar to that with which we started, conseqently we can renumber the L-fuzzy components in such a way that we have $X_{P_{m-1}}=X_{P'_{n-1}}$ and

$$X_{Q_1}\cap X_{Q_2}\cap\cdots\cap X_{Q_{m-2}}=X_{Q'_1}\cap X_{Q'_2}\cap\cdots\cap X_{Q'_{n-2}}.$$

It is now clear that the theorem will be established if we show that m=n. But suppose, for example, that m<n, then after m steps we should obtain

$$(e)=X_{(m)}=X_{\alpha'_1}\cap X_{\alpha'_2}\cap\cdots\cap X_{\alpha'_{n-m}}\subseteq X_{p'_1}\cap X_{p'_2}\cap\cdots\cap X_{p'_{n-m}},$$

which is not possible since all the $X_{p'_j}$ are proper ideals.

Theorem 4.2 shows that if A is a decomposable L-fuzzy ideal then the base sets of the prime L-fuzzy ideals, which are associated with a normal L-fuzzy decomposition of A, depend only on X_A and not on the particular normal L-fuzzy decomposition considered.

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