# HX RING

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In the past few years, the problem of the upgrade of some mathematical structure, from their universes to their power sets, has been paid attention to by more and more people, such as ordered, topological and measurable structure, etc. It is also rather interesting how to consider the upgrade of algebraic structure. In the paper [1], we first studied the upgrade of group, where the concept of HX group has already been advanced. This paper will explore the upgrade of ring, where the concept of HX ring is put forward and some important results are obtained.

## 1. THE DEFINITION OF HX RING AND A STRUCTURAL THEOREM

We always assume that $(R,+,\cdot)$ is a ring in the paper.

In $P_0(R)=2^R-\{\emptyset\}$ we define two algebraic operations:

$$A+B\triangleq\{a+b \mid a\in A,\ b\in B\} \tag{1.1}$$

$$AB\triangleq\{ab \mid a\in A,\ b\in B\} \tag{1.2}$$

It is easy to know that $P_0(R)$ respectively forms a semigroup for the operation (1.1) and (1.2).

It is thing worthy of note that (1.1) and (1.2) does not satisfy the distributive law:

$$A(B+C)=AB+AC, \qquad (B+C)=BA+CA \tag{1.3}$$

But it satisfies the so-called "weak distributive law":

$$A(B+C)\subset AB+AC, \qquad (B+C)A\subset BA+CA \tag{1.4}$$

Let $\mathcal{F}\subset P_0(R)$. $\mathcal{F}$ is called a distributive class on R, if $\mathcal{F}\neq\emptyset$ and for any $A,B,C\in\mathcal{F}$, (1.3) is satisfied. It is easy to see that there must be such the distributive classes.

NOTE: A distributive class may not be closed for the operations (1.1) and (1.2).

Put $\gamma\triangleq\{\mathcal{F} \mid \mathcal{F}$ is a distributive class on $R\}$. Clearly $(\mathcal{F},\subset)$ is a partially ordered set and $\gamma$ must have maximas from Zorn Lemma.

DEFINITION 1.1. Let $\mathcal{R}\in\gamma$. If $\mathcal{R}$ forms a ring for the operations (1.1) and (1.2), then $\mathcal{R}$ is called a HX ring on R, which its null element is denoted by Q. Especially, a HX ring $\mathcal{R}$ is said to be regular, if $0 \in Q$, where 0 is just the null element in R.

We always assume $\mathcal{R}$ is a HX ring on R. Write

$$R^{*}=\cup\{A \mid A\in\mathcal{R}\} \tag{1.4}$$

LEMMA 1.1. 1) (Q,+) is a subsemigroup of R;

2) $QR^{*}\cup R^{*}Q\subset Q$;

3) $0\in R^{*}$ implies $0\in Q$.

Proof. 1) Only to note that 2Q=Q.

2) If $x\in QR^{*}\cup R^{*}Q$, then $x\in QR^{*}$ or $R^{*}Q$. Assuming $x\in QR^{*}$, there exist $a\in Q$ and $b\in R^{*}$, such that x=ab. For $b\in R^{*}$, there exists $A\in\mathcal{R}$, such that $b\in A$, from (1.4), so $x=ab\in QA=Q$.

3) $0\in R^{*}$ implies $(\exists A\in\mathcal{R})(0\in A)$ implies $0\in 0Q\subset R^{*}Q\subset Q$. Q.E.D.

COROLLARY $\mathcal{R}$ is regular iff $0\in R^{*}$.

LEMMA 1.2. 1) $(\forall A\in\mathcal{R})(|A|=|Q|)$;

2) $(\forall A,B\in\mathcal{R})(A\cap B\neq\emptyset$ implies $|A\cap B|=|Q|)$.

Proof. 1) A+Q=A implies $(\forall a\in A)(a+Q\subset A+Q=A)$ implies $|Q|=|a+Q|\leqslant|A|$; $(-A)+A=Q$ implies $(\forall b\in -A)(b+A\subset(-A)+A=Q)$ implies $|A|=|b+A|\leqslant|Q|$.

2) $|A\cap B|\leqslant|A|=Q$; $x\in A\cap B$ implies $x+Q\subset A\cap B$ implies $|Q|=|x+Q|\leqslant|A\cap B|$.

Q.E.D.

Let $I\in P_0(R)$ and H be a subring of R. I is called a semiideal with respect to H, if they satisfy the following conditions:

1) I is a subsemigroup of (R,+);

2) $IH\cup HI\subset I$.

It is easy to learn $0\in I$ from the condition 2).

Let $A\in P_0(R)$, we know that 2A=A $[A^2=A]$ $\Longrightarrow$ A is a subsemigroup of (R,+) $[(R,\cdot)]$; conversely, it is not true. But if $0\in A$ $[e\in A$, when there exists the unit element e of R], then that A is a subsemigroup of $(R,+)[(R,\cdot)]$ iff 2A=A $[A^2=A]$.

From these we have that 2I=I.

LEMMA 1.3. Let I be a semiideal with respect to a subring H of R. If $I^2=I$, then for any $a,b\in H$, we have

$$(a+I)(b+I)=ab+I \tag{1.5}$$

Proof. $(a+I)(b+I)=ab+aI+Ib+I^2=ab+(aI+Ib+I)$. We only need to prove

$$aI+Ib+I=I \tag{1.6}$$

In fact, on the one hand, $aI\cup Ib\subset I$ implies $aI+Ib+I\subset I$; on the other hand, $aI+Ib+I\supset\{a0+0b+c \mid c\in I\}=I$.

Thus (1.6) is true, and so (1.5) be, too. Q.E.D.

LEMMA 1.4. Under the conditions of Lemma 1.3, we have

1) (1.5) has nothing to do with choosing the representative elements;

2) If we put $\mathcal{I} \triangleq \{a+I \mid a \in H\}$, the $\mathcal{I} \in \mathcal{Y}$, i.e. $\mathcal{I}$ is a distributive class on R.

The proof is straightforward.

THEOREM 1.1. Let H be a subring of R, and Q semiideal with respect to H. If $Q^2=Q$, then

$$\mathcal{R} \triangleq \{a+Q \mid a \in H\} \qquad (1.7)$$

is a regular HX ring on R, and Q just the null element of $\mathcal{R}$.

Proof. Noting 2Q=Q, clearly the operation (1.1) is closed in $\mathcal{R}$. The operation (1.2) is also closed in $\mathcal{R}$ from Lemma 1.3. $\mathcal{R}$ is a distributive class from Lemma 1.4. Moreover, since R is a ring it is easy to see that the operations satisfy the associative law.

Making the mapping $f: H \to \mathcal{R}$, $a \longmapsto a+Q$, we have

$$f(a+b)=(a+b)+Q=(a+b)+(Q+Q)=(a+Q)+(b+Q)=f(a)+f(b),$$

$$f(ab)=ab+Q=(a+b)(b+Q)=f(a)f(b)$$

Thus f is surjective homomorphism: $H \sim \mathcal{R}$. So $\mathcal{R}$ is a ring, i,e, $\mathcal{R}$ is a HX ring on R. Noting $0 \in Q$, $\mathcal{R}$ is a regular HX ring.

Moreover, from f(0)=0+Q=Q, Q is the null element of $\mathcal{R}$. Q.E.D.

REMARK: Since $H \sim \mathcal{R}$, $H/\ker f \cong \mathcal{R}$. This means that $\mathcal{R}$ making in this way must isomorphic with the residue class ring of some subring of R. Besides, Theorem 1.1 has the following

COROLLARY 1. $\ker f \subset Q \qquad (1.8)$

Proof. $x \in \ker f$ implies $x+Q=Q$ implies $x=x+0 \in Q$. Q.E.D.

Now we consider how to change (1.8) into equality. Let $\mathcal{R}$ be a regular HX ring. For any $A \in \mathcal{R}$, write

$$\overline{A} \triangleq \{a \in A \mid -a \in -A\} \qquad (1.9)$$

that is said to be the kernel of A. Clearly, $(\forall A,B \in \mathcal{R})(A \subset B \text{ implies } \overline{A} \subset \overline{B})$.

COROLLARY 2. $\overline{\ker f}=\ker f \qquad (1.10)$

Proof. Noting that x+Q=Q implies nx+Q=Q, for any natural number n. So $x \in \ker f$ iff $x+Q=Q$ iff $0+x+Q=Q$ iff $(-x)+2x+Q=Q$ iff $(-x)+Q=Q$ iff $x \in \overline{\ker f}$.
Q.E.D.

COROLLARY 3. $\ker f=\overline{Q} \qquad (1.11)$

Proof. On the one hand, $\overline{\ker f}=\ker f \subset \overline{Q}$. On the other hand, for any $x \in \overline{Q}$, we prove that x+Q=Q. That $x+Q \subset Q$ is clear; conversely, for any $y \in Q$, $y=y+0=y+x+(-x)=x+[y+(-x)]$. Since $x \in \overline{Q}$, so $y+(-x) \in Q$, thus $y \in x+Q$, i.e. x+Q=Q. Hence $\overline{Q} \subset \ker f$. Q.E.D.

COROLLARY 4. $\ker f=Q$ iff $Q=\overline{Q}$

## 2. QUSI RESIDUE CLASS RING

Theorem 1.1 means that we can form a regular HX ring $\mathcal{R}$ by using a subring H of R and semiideal with respect to H and $\mathcal{R}$ presents the form of coset so that Q just is the null element of $\mathcal{R}$. As the inversion of Theorem 1.1, we have the following

PROBLEM: Let $\mathcal{R}$ be a regular HX ring on R. Whether are there a subring H and a semiideal I with respect to H such that $\mathcal{R}=\{a+I \mid a\in H\}$ and Q=I?

If $\mathcal{R}$ is regular, write

$$\overline{R}=\cup\{\overline{A} \mid A\in\mathcal{R}\} \tag{2.1}$$

Clearly $(\forall A\in\mathcal{R})(\overline{A}\neq\emptyset)$, so $\overline{R}\neq\emptyset$.

THEOREM 2.1. If $\mathcal{R}$ is a regular HX ring on R, then

1) $\overline{R}$ is a subring of R;

2) Q is a semiideal with respect to R;

3) $\mathcal{R}=\{a+Q \mid a\in\overline{R}\}$.

Proof. 1) $\forall a,b\in\overline{R}$, $\exists A,B\in\mathcal{R}$, such that $a\in\overline{A}$, $b\in\overline{B}$. First prove that $a-b\in\overline{R}$. Since $a-b=a+(-b)\in A+(-B)=C\in\mathcal{R}$ and noting $-[a+(-b)]=b+(-a)\in B+(-A)=-C$, so $a-b\in\overline{C}\subset\overline{R}$. Now we prove that $ab\in\overline{R}$. For this we should prove a fact: for any $A,B\in\mathcal{R}$, must have the following

$$\overline{A}\,\overline{B}\subset\overline{AB} \tag{2.2}$$

$x\in\overline{A}\,\overline{B}$ implies $(\exists a\in\overline{A})(\exists b\in\overline{B})(x=ab)$ implies $-x=-(ab)=(-a)b\in(-A)B=-(AB)$ implies $x\in\overline{AB}$, so (2.2) is true.

By the (2.2) we have $ab\in\overline{A}\,\overline{B}\subset\overline{AB}\subset\overline{R}$.

In a ward, $\overline{R}$ is a ring.

2) Noting $Q\overline{R}\cup\overline{R}Q\subset QR^{*}\cup R^{*}Q$, Q is the semiideal with respect to $\overline{R}$ from Theorem 1.1.

3) For any $A\in\mathcal{R}$, taking $a\in\overline{A}$, we may prove that $a+Q=A$. On the one hand, $a+Q\subset A+Q=A$. On the other hand, $b\in A$ implies $b=0+b=(a-a)+b=a+(b-a)=a+(A-A)=a+Q$, so $A\subset a+Q$. Thus $A=a+Q$. Therefore $\mathcal{R}\subset\{a+Q \mid a\in\overline{R}\}$.

Conversely, for any $a\in\overline{R}$, there exists $A\in\mathcal{R}$, such that $a\in\overline{A}$. So $A=a+Q$ from above the process of the proof, i.e., $\{a+Q \mid a\in\overline{R}\}\subset\mathcal{R}$.

In a ward, $\mathcal{R}=\{a+Q \mid a\in\overline{R}\}$. Q.E.D.

Theorem 1.1 has answered the problem above in the affirmative.

DEFINITION 2.1. Let Q be a semiideal with respect to R. The regular HX ring making as follows

$$R|Q\triangleq\{a+Q \mid a\in R\} \tag{2.3}$$

is called qusi residue class ring which R is with respect to Q.

According as Definition 2.1, if $\mathcal{R}$ is a regular HX ring on R, then $\mathcal{R}$ must be the qusi residue class ring which R is with respect to some subring of R:

$$\mathcal{R} = \overline{R}|Q \qquad (2.4)$$

Theorem 2.1 has also the following

COROLLARY $(\forall A\in\mathcal{R})(a\in\overline{A}$ implies $a+Q=A)$

Under stronger conditions Theorem 1.1 is the following form:

THEOREM 2.2. Let $\mathcal{R}$ be a HX ring on R. If Q is a ideal of R, then

1) $\mathcal{R}=\{a+Q|\ a\in R^*\}$;

2) $R^*$ is a subring of R;

3) $\mathcal{R}=R^*/Q$.

Proof. 1) $\forall A\in\mathcal{R}$, taking $a\in A$, then $a+Q\subset A+Q=Q$. It can be proved that $a+Q=A$. If it is not true, then there exists $b\in A-(a+Q)$. We can prove that $b-a \notin Q$. In fact, if $b-a=c\in Q$, then $b=a+c\in a+Q$. This is in contradication with be $a+Q$. Taking $d\in -A$, we have $a+d, a+b\in A+(-A)=Q$. So $b-a=b+d-d-a=(b+d)+[-(a+d)]\in Q$. This is in contradication with $b-a\notin Q$. Thus $a+Q=A$, i.e. $\mathcal{R}\subset\{a+Q|\ a\in R^*\}$.

Conversely, for any $a\in R^*$, there exists $A\in\mathcal{R}$, such that $a\in A$. So $a+Q=A\in\mathcal{R}$. Thus $\{a+Q|\ a\in R^*\}\subset\mathcal{R}$.

In a ward, $\mathcal{R}=\{a+Q|\ a\in R^*\}$.

2) $\forall a\in R^*$, there is $A\in\mathcal{R}$, such that $a\in A$. Noting $0\in Q$ and $A+(-A)=Q$, we know that, there are $b\in A, -b\in -A$, such that $b-b=0$. From $A=b+Q$, there is $c\in Q$, such that $a=b+c$. So $-a=-(b+c)=(-c)+(-b)\in Q+(-A)=-A\subset R^*$. Thus $(R^*,+)$ is a subgroup of R.

Moreover, $\forall a,b\in R^*$, $\exists A,B\in\mathcal{R}$, such that $a\in A, b\in B$. So $ab\in AB\subset R^*$. Thus $(R^*,\cdot)$ is a subsemigroup of R.

In a ward, $R^*$ is a subring of R.

3) Clearly Q is a ideal of $R^*$. So $\mathcal{R}=R^*/Q$. Q.E.D.

## 3. THE RELATION BETWEEN QUSI RESIDUE CLASS RING AND RESIDUE CLASS RING

LEMMA 3.1. If $\mathcal{R}$ is regular, then $\overline{Q}$ is the ideal of $\overline{R}$.

Proof. Making the mapping $f:\overline{R}\longrightarrow\mathcal{R}$, $a\longmapsto a+Q$, clearly f is a surjective homomorphism. From the corillary 3 in the section 3, $\ker f=\overline{Q}$. So $\overline{Q}$ is an ideal of $\overline{R}$. Q.E.D.

THEOREM 3.1. If $\mathcal{R}$ is a regular HX ring on R, then

$$\overline{R}/\overline{Q} \cong \overline{R}|Q \qquad (3.1)$$

Proof. From the surjective homomorphism $f:\overline{R}\longrightarrow\overline{R}|Q$, $a\longmapsto a+Q$, we have

$\overline{R}/\ker f \cong \overline{R}|Q$. And by the corollary in the section, (3.1) is true. Q.E.D.

Let $\mathcal{R}$ a regular HX ring, Write

$$\overline{\mathcal{R}}=\{\overline{A} \mid A\in\mathcal{R}\} \tag{3.2}$$

LEMMA 3.2. $(\forall\overline{A}\in\overline{\mathcal{R}})(\forall a\in\overline{A})(\overline{a+Q}=a+\overline{Q})$

Proof. $x\in\overline{a+Q}$ implies $(\exists\, b\in Q)(x=a+b)$ implies $-b=(-x)+a\in[-(a+Q)]+A=(-A)+A=Q$ implies $b\in\overline{Q}$ implies $x=a+b\in a+\overline{Q}$;

$x\in a+\overline{Q}$ implies $(\exists\, b\in\overline{Q})(x=a+b)$ implies $-x=(-a)+(-b)\in-(a+Q)$ implies $x\in\overline{a+Q}$.

In a ward, $\overline{a+Q}=a+\overline{Q}$. Q.E.D.

THEOREM 3.2. If $\mathcal{R}$ is a regular HX ring, then

$$\overline{\mathcal{R}}=\overline{R}/\overline{Q}. \tag{3.3}$$

Proof. $\forall\,\overline{A}\in\overline{\mathcal{R}}$, taking $a\in\overline{A}$, $a+Q=A$ from the corollary of Theorem 2.1. Thus $\overline{A}=\overline{a+Q}$. Noting Lemma 3.2, we have $\overline{A}=a+\overline{Q}\in\overline{R}/\overline{Q}$. So $\overline{\mathcal{R}}\subset\overline{R}/\overline{Q}$.

Conversely, $\forall\, a+\overline{Q}\in\overline{R}/\overline{Q}$, since $a\in\overline{R}$, then $\exists\,\overline{A}\in\overline{\mathcal{R}}$, such that $a\in\overline{A}$. From the process of above proof, we learn that $a+\overline{Q}=\overline{a+Q}=\overline{A}\in\overline{\mathcal{R}}$. So $\overline{R}/\overline{Q}\subset\overline{\mathcal{R}}$.

In a ward, $\overline{\mathcal{R}}=\overline{R}/\overline{Q}$. Q.E.D.

COROLLARY If $\mathcal{R}$ is regular, then

$$(\forall A,B\in\mathcal{R})(\overline{A}\,\overline{B}=\overline{AB}) \tag{3.4}$$

Proof. From (2.2) $\overline{A}\,\overline{B}\subset\overline{AB}$. And from (3.3) we have $\overline{A}\,\overline{B}=\overline{AB}$. Q.E.D.

THEOREM 3.3. Let f be a surjective homomorphism from R to another ring R'. We have

1) If $\mathcal{R}$ is a HX ring on R, then

$$\mathcal{R}'\triangleq\{f(A) \mid A\in\mathcal{R}\} \tag{3.5}$$

is a HX ring on R', and $\mathcal{R}\sim\mathcal{R}'$;

2) If $\mathcal{R}'$ is a HX ring on R', then

$$\mathcal{R}\triangleq\{f^{-1}(A') \mid A'\in\mathcal{R}'\} \tag{3.6}$$

is a HX ring on R, and $\mathcal{R}\sim\mathcal{R}'$.

The proof is straightforward.

## REFERENCE

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[3] Carl Faith, Algebra II - Ring Theory, Springer-Verlag, Berling Heidelberg New York, 1976.