Book review of

The Mathematics of Fuzzy Systems

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There are not so many books devoted to the mathematical aspects of fuzzy set theory. In fact the most recent one is Negoita and Ralescu 1975 monograph [1]. This new book is a collection of papers most of which share the common concern of putting concepts of approximation into formal settings, that is logical systems or categorial ones. It is mainly based on contributions to a workshop held in Napoli in 1985.

The early eighties have witnessed a significant breakthrough in the foundations of fuzzy set theory from various point of views including algebra, category theory, measure theory and logic. From an algebraic point of view, the basic role played by special classes of semi-groups on the unit interval such as triangular norms an co-norms, in the definition of fuzzy set theoretic operations has been acknowledged. These results are extensively covered by Gottwald and Yager's contributions. Gottwald, provides a careful review of t-norm-based logics with involutive negation, and intuitionistic implication defined from conjunction via a residuation-like operation. Then he considers extension of Zadeh's similarity relations and fuzzy orderings with triangular norms. The latter results parallel those obtained by Trillas, Valverde [2] and other members of the Barcelona School on indistinguishibility relations, although they are not mentioned in the paper. Yager considers the generalizations of other concepts with triangular norms, namely the extension principle, and various measures of fuzzy events.

One of the basic debates in the logical foundations of fuzzy sets is the choice of a suitable structure for the set of truth values. The dilemma is between. De Morgan algebras, where negation is involutive but the law of contradiction ($A \cap \bar{A} = \emptyset$) may fail to hold, and intuitionistic algebras where negation is not involutive but where the law of contradiction holds. Zadeh's original fuzzy set theory (the max-min system) is a De Morgan algebra, but Zadeh's definition of fuzzy set inclusion is consistent with intuitionistic implication, and not with the Boolean-like definition ($\bar{A} \cup B$). This problem is informally discussed by Muir, from the stand-point of a comparison with probabilility theory. Gottwald t-norm logic follows the spirit of Zadeh's construct, i.e. De Morgan algebras. Contrastedly, several systems of fuzzy logics more in the spirit of intuitionistic theories are presented in the book.

Takeuti and Titani overview their work on a max-min intuitionistic system which uses Gödel implication, and defines negation from implication in the usual way (A → False). Cattaneo and Nistico consider an hybrid system with two negations, one that is involutive and one that is intuitionistic. Combining these negations leads to modal operators of possibility (◊) and necessity (□) that enable a modal-logic-like approach to this system to be contemplated. Semantics of such a logical system are considered. A proposition p in this logic can be viewed as a pair of disjoint sets corresponding to □p and the negation of ◊p respectively. We thus come close to Pawlak's rough sets and the modal logic which underlies them, which is reviewed by Fariñas del Cerro and Prade. These authors extend rough sets to the case when the equivalence relation which expresses indiscernibility becomes a fuzzy similarity relation, and upper and lower approximations of fuzzy sets are derived. A pair of fuzzy sets is obtained this way. Another construct of the same kind, called twofold fuzzy set, is compared with fuzzy rough sets, twofold fuzzy sets arise in a quite different context, namely when computing a set of objects which must satisfy a clear-cut property when the corresponding attribute values are ill-known, and described by possibility distributions.

With rough sets, we come close to the idea of approximation due to indiscernibility, described by a reflexive and symmetric fuzzy relation on a set. These indiscernibility relations are also central in the contributions by Katz, Cerruti and Höhle, and Bernard. Katz interprets indiscernibility as indifference in the scope of choice theory. He casts results drawn from measurement theory, psychometry and decision theory, into a many-valued logical setting, namely the one of Lukasiewicz logic, in order to capture inconsistencies in choice behavior. Axioms for indifference relations, gain functions, preference and between-ness relation are studied. Cerruti and Höhle describe the category LUS of sets equipped with a fuzzy equality relation with a general transitivity property (based on triangular-norm-like operations). Fuzzy sets emerge in this setting as morphisms, and the internal logic, genralizes Zadeh's original fuzzy sets. LUS contrasts with other types of categorical constructs which imbed intuitionistic types of logic, such as Higgs' category [6]. Bernard puts together fuzzy sets and Scott's domains into a new concept he calls "multi-ensembles" for which he provides the proper categorial setting. Gerla introduces fuzzy models and defines a categorial setting for them. All the above-mentioned papers are somehow related each other. It makes their reading very stimulating.

The paper by Muir contains other discussions related to the modeling of fuzziness, as opposed to the representation of aleatory phenomena with a frequentist interpretation. He suggests to relax the truth functionality assumption of many-valued logics by means of conditional membership values related to the idea of many-valued implication. He argues in favor of a qualitative view of fuzzy sets, and requires that membership functions be replaced by membership relations, each expressing an ordering of objects in terms of their compatibility with a given fuzzy concept. Muir also emphasizes the important rôle of lattices in fuzzy set theory and suggests a lattice-valued probability theory. These ideas are further elaborated in a companion paper by Warner and Muir.

The book also includes two contributions on fuzzy random variables, one by A. Ralescu and D. Ralescu on limit theorems, and one by Kruse, on a genralization of the variance. These two contributions which are summaries of research reflect the importance of fuzzy random variables as a newly emerged field of investigations at the interface between probability theory and fuzzy set theory. Two recent monographs on this topic, by Kruse [4] and Negoita and Ralescu [5] are available.

The editors of the volume also contribute an interesting paper on measures of fuzziness in a lattice, thus pursuing preliminary works by De Luca, Termini and Yager. Another stand-alone (but surprizing) contribution is that of Speranza on a fuzzy extension of projective geometry.

On the whole, and if we except a few contributions only loosely related to fuzzy sets, this collection of papers is a very good mirror of the on- going activity in the mathematics of fuzzy logic ; the required level for easy reading is not the same for all the papers ; some are very sophisticated and may be useful only to pure mathematicians. But the overall quality of this volume is unusually high and the contents are unusually coherent for such a kind of book, although the links between the papers are not obvious at first glance due to an alphabetical ordering of the contributions. One cannot but recommend the reading of this edited volume, as a stimulating source of new ideas for future research in the logics of vagueness and approximation.

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References

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