EXTENDED FUZZY PREFERENCES (I): RANDOMIZED EXTENSION

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Abstract: This paper deals with decision-making problems where a fuzzy preference relation with no unfuzzy nondominated alternatives has been defined. Two types of extended problems (randomized and fuzzified) are considered in order to get a solution. In this first part randomized extended fuzzy relations are analyzed, and its connection with Fishburn's SSB utility theory is stablished.

Keywords: Fuzzy Preference Relation, Decision Making, Nondominated Alternatives, SSB Utility Theory.

1. Introduction

Since we often do not have fully clear idea about preferences between alternatives in a given finite set of feasible alternatives X, such preferences can be modellized in many cases by a fuzzy preference relation. We shall suppose that much a fuzzy preference relation has been evaluated by an expert, in such a way that a decision problem (X,μ) is defined, μ being a mapping $\mu: X \times X \rightarrow |0,1|$, and $\mu(x,y)$ meaning the degree to which alternative x is not worse than alternative y. This fuzzy preference relation μ will be assumed to be reflexive (i.e., $\mu(x,x) = 1 \quad \forall x \in X$) through out the paper. Our problem is how a reasonable choice set must be defined, on the basis of such a fuzzy pairwise comparisions.

A concept of solution for the above decision-making problem was introduced by Orlovsky |9|, by considering the fuzzy subset of nondominated alternatives. Its membership function is given by $\mu^{ND}(x) = 1 - \max_{y \in X} \mu^{S}(y,x) \quad \forall x \in X$, where $\mu^{S}(y,x) = \max(\mu(y,x) - \mu(x,y),0)$ means the degree to which alternative x is strictly dominated by alternative y. Therefore, the value $\mu^{ND}(x)$ can be understood as the degree to which alternative x is dominated by no one alternative. The fuzzy subset of nondominated alternatives plays the role of a fuzzy subset of solutions: the greater the degree of nondomination of an alternative, the better such an alternative.

It seems natural that any reasonable choice must be inside the set of unfuzzy nondominated alternatives

$$X^{UND} = \{x \in X/\mu^{ND}(x) = 1\} =$$
$$= \{x \in X/\mu(x,y) \geq \mu(y,x) \quad \forall y \in X\}$$

in case of being non-empty. Hence, it will be of special interest the stablishment of sufficient conditions assuring the non-emptiness of X^{UND}. For example, Orlovsky |9| proved that $X^{UND} \neq \emptyset$ when μ verifies max-min transitivity ($\mu(x,y) \geq \min(\mu(x,z),\mu(z,y)) \quad \forall x,y,z \in X$), and Montero-Tejada |7| showed that $Y^{UND} \neq \emptyset \quad \forall Y \subset X$ if and only if an associated crisp binary relation was acyclic in the sense of Sen |12| (the fact that the set of unfuzzy nondominated alternatives can be viewed as the choice set of a crisp preference relation has been also pointed out by Zhukovin et al. |13|).

Some results related to decision-making under the Orlovsky's concept of solution can be seen in Kolodziejczyk |4| and Orlovsky |11|. But new concepts of solution are needed when there is no unfuzzy nondominated alternative. Some possible choice sets proposed in the past has been collected in Montero-Tejada |6|, together with other choice sets which can be defined only through successive discarding methods.

In this paper, two distinct alternatives will be discussed: i) randomization, and ii) fuzzification. Both problems can be formulated as follows: i) can we define a reasonable fuzzy preference relation between probability distributions on X, with a non-empty set of unfuzzy nondominated distributions?, and ii) can we define a reasonable fuzzy preference relation between fuzzy subsets in X, with a non-empty set of unfuzzy nondominated fuzzy subsets?. The first question has been solved positively in Montero-Tejada |5|, by considering a linear extension which has a deep link with Fishburn's SSB utility theory (see |1|). The problem of how to define an extended fuzzy preference relation between fuzzy subsets has been initiated by Orlovsky |10|.

2. Randomized extension

Let us consider $X = \{x_1,\ldots,x_n\}$ and the set of probability distributions over X,

$$\hat{X} = \{p = (p_1,\ldots,p_n)^t \in \mathbb{R}^n / \sum_{i=1}^{n} p_i = 1 \ , \ p_i \geq 0 \quad \forall i\}$$

Given a fuzzy preference μ over X, it can be represented by a $n \times n$ matrix R such that $\mu(x_i,x_j)$ is the element in file i and column j. We shall denote $\mu_{ij} = \mu(x_i,x_j)$.

As pointed out above, one can try to solve the decision problem by considering a randomized extended problem $(\hat{X},\hat{\mu})$, where $\hat{\mu}$ must be an appropiated fuzzy preference relation defined over the set of probability distributions $\hat{X}$. In other words, a mapping

$$\hat{\mu} : \hat{X} \times \hat{X} \to |0,1|$$

must be defined, $\mu(p,q)$ meaning the degree of preference for distribution p over distribution q.

Let us denote $d^i \in \hat{X}$ the degenerate distribution over alternative x_i ($d^i_j = 0 \quad \forall j \neq i$, $d^i_i = 1$). Then the following definition makes sense:

Definition1.- Given a fuzzy preference relation μ over X, any fuzzy preference relation $\hat{\mu}$ over $\hat{X}$ is said to be a randomized extension if $\hat{\mu}(d^i,d^j) = \mu_{ij} \quad \forall i,j$.

Following Orlovsky's idea, when an appropiate extended problem $(\hat{X},\hat{\mu})$ has been defined, strict preferences between probability distributions can be represented by a fuzzy preference relation with membership function $\hat{\mu}^S(p,q) = \max(\hat{\mu}(p,q) - \hat{\mu}(q,p),o)$ in such a way that the fuzzy subset of nondominated alternatives is given by

$$\hat{\mu}^{ND}(p) = 1 - \sup_{q \in \hat{X}} \hat{\mu}^S(q,p)$$

Therefore, the set of unfuzzy nondominated distributions can be written as

$$\begin{aligned} \hat{X}^{UND} &= \{p \in \hat{X} / \hat{\mu}^{ND}(p) = 1\} = \\ &= \{p \in \hat{X} / \hat{\mu}(p,q) \geq \hat{\mu}(q,p) \quad \forall q \in \hat{X}\} \end{aligned}$$

Any appropiate randomized extension must verify that a degenerate distribution p^i is nondominated if and only if x_i is a nondominated alternative. Moreover, randomization will be useful in a wide sense when the existence of unfuzzy nondominated distributions is assured, even if there is no unfuzzy nondominated alternative.

In any case, when a probability distribution $p = (p_1,...,p_n)$ is supposed, each alternative x_i is chosen with probability p_i. Hence, it seems natural to assume that mathematical expectation represents a basis for the comparision between probability distribution. In particular, the following conditions for randomized extensions can be accepted by the decision-makes:

$$\text{R1)}\quad \hat{\mu}(p,q) = \sum_{i=1}^{n} p_i \cdot \hat{\mu}(d^i,q)$$

$$\text{R2)}\quad \hat{\mu}(p,q) = \sum_{j=1}^{n} q_j \cdot \hat{\mu}(p,d^j)$$

If these conditions are assumed, it is clear that $\hat{\mu}$ must be the mathematical expectation of μ with respect to the product probability $p \times q$:

$$\hat{\mu}(p,q) = \sum_i p_i\ \hat{\mu}(d^i,q) = \sum_i p_i \sum_j q_j\ \hat{\mu}(d^i,d^j) =$$

$$= \sum_{i,j} p_i\ q_j\ \mu(x_i,x_j) = p^t Rq$$

This is the "linear extension" proposed in Montero-Tejada |5|. We shall denote $\hat{\mu}_L(p,q) = \sum_{i,j} p_i q_j\ \mu_{ij}$, and it represents the expected degree of preference of distribution p over distribution q. Trivially, $\hat{\mu}_L(d^i,d^j) = \mu_{ij}$, and denoting

$$\hat{\mu}_L^s(p,q) = \max(\hat{\mu}_L(p,q) - \hat{\mu}_L(q,p),o)$$

$$\hat{\mu}_L^{ND}(p) = 1 - \sup_{q \in \hat{X}} \hat{\mu}_L^s(q,p)$$

$$\hat{X}_L^{UND} = \{p \in \hat{X} / p^t Rq \geq q^t Rp \quad \forall q \in \hat{X}\} =$$

$$= \{p \in \hat{X} / p^t (R-R^t)\ q \geq 0 \quad \forall q \in \hat{X}\}$$

it is easy to see that $x_k \in X^{UND}$ if and only if $d^k \in \hat{X}_L^{UND}$. We shall denote $\emptyset_L(p,q) = p^t(R-R^t)q$, and it is clear that $\emptyset_L(p,q) = -\emptyset(q,p)$, since $R-R^t = -(R-R^t)^t$. The main result was proved by applying classical Matrix Game Theory (unfuzzy nondominated distributions are obtained as the solution of a matriz game):

Teorem 1 |5|.- $\hat{X}_L^{UND}$ is convex, compact and non-empty.

Therefore, $\hat{X}_L^{UND} \neq \emptyset$ in any case, whithout impossing any condition. Since the linear extension seems to be a natural preference between probability distributions, unfuzzy nondominated distributions can be considered in decision-making problems (X,μ) with no unfuzzy nondominated alternatives. Unfuzzy nondominated distributions can be obtained as the solution of a mathematical linear programming problem.

3. SSB utility theory

In this section, the connection between the SSB utility theory introduced by Fishburn |1| and the previous linear extension is studied.

SSB utility theory is a generalization of the linear utility theory of von Neumann and Morgenstern |8|. As pointed out by Fishburn |3| it retains the implications of continuity, convexity monotonicity and simmetry, but drops transitivity and independence axioms, in such a way that it will be compatible with the phenomena of strict preference cycles, preference reversals, and violations of independence.

SSB utility theory represents preferences between probability distributions by a skew-symmetric bilinear functional defined on pairs of distributions (the designation "SSB" is based upon such a numerical representation). Hence, preferences are represented by a bivariate rather than an univariate real valued function.

Given a convex set P of probability measures defined on a Boolean σ-algebra $\Delta (\lambda p+(1-\lambda)q \in P \quad \forall p,q \in P \ , \quad \forall\lambda \in (0,1)$, where $(\lambda p+(1-\lambda)q)(A) = \lambda p(A)+(1-\lambda)q(A) \quad \forall A \in \Delta)$, a binary preference relation $\geq$ on P satisfies SSB utility theory if there exists a skew-symmetric function $\emptyset : P \times P \to \mathbb{R}$, being linear in each argument, such that $p \geq q$ if and only if $\emptyset(p,q) \geq 0$ (as usual, $p > q$ if $p \geq q$ but not $q \geq p$, and $p \sim q$ if $p \geq q$ and $q \geq p$). Skew-symmetry means that $\emptyset(q,p) = -\emptyset(q,p)$ for all $p,q \in P$, and linearity means that

$$\emptyset(\lambda p + (1-\lambda)q,\tau) = \lambda\emptyset(p,\tau) + (1-\lambda)\ \emptyset(q,\tau)$$
$$\emptyset(\tau,\lambda p + (1-\lambda)q) = \lambda\emptyset(\tau,p) + (1-\lambda)\ \emptyset(\tau,q)$$

for all $p,q,\tau \in P$ and $\lambda \in |0,1|$. It must be pointed out that linearity in one argument is deduced from the linearity in the other argument if skew-symmetry holds. $\emptyset$ is said to be a SSB representation of $\geqq$.

When P is the set of all probability distributions on the finite set X($P = \hat{X}$ in our notation, and Δ the set of all subsets in X), bilinearity means that

$$\emptyset(p,q) = \sum_{i,j} p_i q_j \ \emptyset(d^i,d^j)$$

(the integral form is characterized in |2|, when X is not finite).

Fishburn |1| himself suggests that values $\emptyset(x_i,x_j)$ can be viewed as some measure of intensity for alternative x_i over alternative x_j, but in a later paper |3| he points out that the use of the language "preference intensity" is no more than suggestive: SSB utility function can be viewed merely as a convenient vehicle to represent qualitative preferences. This difficulty of meaning can be easily avoided in the fuzzy set theory framework.

Since the idea of intensity of preference is more appropiated to randomized extension than to SSB representation, the following definition is suggested:

Definition 2.- Let μ be a fuzzy preference relation defined on a set Y. Then the mapping

$$\emptyset : Y \times Y \to (-1,1)$$

such that $\emptyset(a,b) = \mu(a,b) - \mu(b,a)$ is said to be the basic representation of μ .

Therefore $\emptyset_L$ function is the basic representation of the fuzzy preference relation $\hat{\mu}_L$ defined on $\hat{X}$. From a mathematical point of view, $\emptyset_L$ could be a special SSB utility function, with $\emptyset_L(d^i,d^j) = \mu_{ij} - \mu_{ji}$. But linear extension $\hat{\mu}_L$ is the basis for a SSB utility function $\emptyset_L$ only if a subyacent preference relation $\geqq$ on X is supposed to be defined ($p \geqq q$ if $\emptyset_L(p,q) \geqq 0$). In this way, SSB model and linear extension are quite different: linear extension defines normatively such a preference relation, whereas SSB model is a quantitative representation of a preference relation being "rational" in the sense of conditions C,D and S, as given in the following result :

Theorem 1 |1|.- Let P be a non empty convex set of probability measures. Then there is a SSB representation for a preference relation $\geq$ on P, if and only if the following axioms are satisfied:

C) if $p > q$ and $q > \tau$, then $q \sim \alpha p + (1-\alpha)\tau$ for at least one $0 < \alpha < 1$ (continuity).

D) if $p > q$ and $p \geq \tau$, then $p > \lambda q + (1-\lambda)\tau$ $\forall \lambda \in (0,1)$; if $q > p$ and $\tau \geq p$, then $\lambda q + (1-\lambda)\tau > p$; of $p \sim q$ and $p \sim \tau$, then $p \sim \lambda q + (1-\lambda)\tau$ (dominance).

S) if $p > q > \tau$, $p > \tau$ and $q \sim \frac{1}{2} p + \frac{1}{2} \tau$, then $\lambda p + (1-\lambda)\tau \sim$ $\sim \frac{1}{2} p + \frac{1}{2} q$ if and only if $\lambda \tau + (1-\lambda)p \sim \frac{1}{2} \tau + \frac{1}{2} q$ (symmetry).

Moreover, it is proved that $\emptyset$ is unique up to a similarity transformation (i.e., $\emptyset(p,q) \geq 0$ if and only if $a\emptyset(p,q) \geq 0$ $\forall a > 0$, and therefore $a \cdot \emptyset$ is also a SSB representation).

If fuzzy set theory gives a natural intuitionism of the SSB model, it is clear that the SSB model gives a mathematical background for the linear extension of fuzzy preference relations. Many properties of SSB utility representation can be usefull in the analysis of the linear extension. Moreover, if $\emptyset$ is a SSB utility function being the basic representation of a fuzzy preference relation $\hat{\mu}$ on X , then $\hat{\mu}$ is the linear extension of a fuzzy preference μ on X : it is enough to define μ such that $\mu^s(x_i,x_j) = \max\ (\emptyset(d^i,d^j),o)$.

Note that a SSB utility function $\emptyset$ is a basic representation of some fuzzy preference relation on $\hat{X}$ if an only if $|\emptyset(p,q)| \leq 1$ $\forall p,q \in \hat{X}$.

In any case, the choice set is given by the set of distributions p such that $\emptyset(p,q) \geq 0$ $\forall q \in \hat{X}$, where $\emptyset$ is a SSB representation of a binary preference $\geq$, or the basic representation of a fuzzy preference relation $\hat{\mu}$. But two difficulties arise when both approximations are applied: on one hand, randomization is often rejected by decision-maker; on the other hand, as pointed out by Fishburn (see |2|), disturbing implications appears in real problems (usual intuition fails in their justification, mainly due to the non consideration of long cycles effect).

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