ON THE STATES OF A DYNAMICAL SYSTEM UNDER FUZZY CONTROL

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CONSIDER A DYNAMICAL SYSTEM PROVIDED WITH A PROGRAMMED CONTROL. IT IS CLEAR THAT BECAUSE OF PERTURBING INFLUENCES THE RESULTANT BEHAVIOUR OF THE SYSTEM MIGHT DIFFER FROM THE PROGRAMMED ONE AND SO OUR KNOWLEDGE OF THE SYSTEM'S STATE MIGHT DECREASE IN TIME EVEN IF THE KNOWLEDGE AT THE INITIAL INSTANT BE EXHAUSTIVE. OUR FIRST AIM IS TO ESTIMATE SUCH A PROGRESSIVE UNCERTAINTY IN TERMS OF FUZZY STATES FOR A LINEAR CONTINUOUS SYSTEM (THEOREM 1). THEN LET'S NOT INSIST THAT THE KNOWLEDGE AT THE INITIAL INSTANT BE EXHAUSTIVE. LET, INSTEAD, THE KNOWLEDGE INCREASE, RATHER THAN DECREASE, IN TIME. OUR SECOND AIM IS TO ESTIMATE SUCH A REGRESSIVE UNCERTAINTY (THEOREM 2).

AS A NAIVE ILLUSTRATION, CONSIDER THE PROBLEM OF DEMAND PROGRAMMING. IT IS CLEAR THAT VARIATIONS IN CONJUNCTURE DEPRECIATE EXCESSIVE KNOWLEDGE OF THE INITIAL DEMAND. ON THE OTHER HAND, INSUFFICIENT KNOWLEDGE IS UNDESIRABLE TOO. SO A COMPROMISE BODY OF KNOWLEDGE IS NECESSARY.

TO APPLY THE ESTIMATES OBTAINED HERE, IN COMPARISON WITH (CRISP) ELLIPSOIDAL ESTIMATES [1], ONE NEEDN'T SOLVE ANY CAUCHY PROBLEMS. YET ONE HAS TO ESTIMATE CERTAIN SPECTRAL CHARACTERISTICS OF THE SYSTEM. AS REGARDS THE DISTURBANCE IN CONTROL, WE SUPPOSE THAT THE SET OF ITS POSSIBLE VALUES, VIZ. THE FUZZY SET OF ITS VALUES [2], INCREASES IN TIME.

1. THE STATE SPACE AND FUZZY-VALUED FUNCTIONS OF TIME

LET x, y, x_0 BE POINTS OF THE n-DIMENSIONAL EUCLIDEAN SPASE, AND $\tilde{x}$, $\tilde{y}$, $\tilde{x}_0$ ITS NONEMPTY BOUNDED FUZZY SUBSETS (IN THE SENSE OF BEING NORMALIZED IN HEIGHT AND BOUNDED IN SUPPORT [3]). BY TRADITION, FUZZY SETS ARE REPLACED WITH THEIR LEVEL SETS, AND CRISP ONES ARE IDENTIFIED WITH THEIR SUPPORTS. LET $0<\lambda<1$, AND lev_λ DESIGNATE LEVEL SETS. LET τ, θ, τ_0 BE NON-NEGATIVE REAL NUMBERS. LET A BE A REAL n-BY-n MATRIX WITH THE TRANSPOSED MATRIX A^*, THE MATRIX $[A]=(A+A^*)/2$, THE INVERSE MATRIX A^{-1} (IF IT EXISTS), THE EXPONENTIAL e^A AND THE SPECTRAL NORM $\|A\|$. THE UNIT n-BY-n MATRIX IS DENOTED E.

ON THE COLLECTION OF ALL $\tilde{x}$'S LET US DEFINE THE ORDER BY INCLUSION $\subseteq$, THE MINKOWSKI ADDITION $+$, THE MULTIPLICATION BY A, AND THE (WEIGHTED) DIAMETER $diam$, IMPLYING THAT

$$\tilde{x}\subseteq\tilde{y} \Leftrightarrow (x\in lev_\lambda\tilde{x} \Rightarrow x\in lev_\lambda\tilde{y}\ \forall x)\ \forall\lambda,$$

$$lev_\lambda(\tilde{x}+\tilde{y})=\{x+y:\ x\in lev_\lambda\tilde{x},\ y\in lev_\lambda\tilde{y}\},$$

$$lev_\lambda(A\tilde{x})=\{Ax:\ x\in lev_\lambda\tilde{x}\},$$

$$diam\,\tilde{x}=diam_\omega\tilde{x}=\sup_\lambda\omega_\lambda\sup\{|x-y|:x,y\in lev_\lambda\tilde{x}\},$$

WHERE THE FUNCTION $\lambda\to\omega_\lambda\in[0,1]$ IS A WEIGHT TO LEVEL SETS. IT IS EASY TO REVEAL A MUTUAL SUBORDINATION WITHIN THIS STRUCTURE.

IN THE FOLLOWING THE FUNCTIONAL diam WILL PLAY A SPECIAL ROLE, SO LET US EXHIBIT ITS ATTRIBUTES:

$$0 \leq \operatorname{diam} \tilde{x} < \infty, \qquad \operatorname{diam} \tilde{x} = 0 \Leftrightarrow \tilde{x} = \{x\},$$

$$\tilde{x} \subseteq \tilde{y} \Rightarrow \operatorname{diam} \tilde{x} \leq \operatorname{diam} \tilde{y},$$

$$\max\{\operatorname{diam} \tilde{x}, \operatorname{diam} \tilde{y}\} \leq \operatorname{diam}(\tilde{x}+\tilde{y}) \leq \operatorname{diam} \tilde{x} + \operatorname{diam} \tilde{y},$$

$$\frac{\operatorname{diam} \tilde{x}}{\|A^{-1}\|} \leq \operatorname{diam} A\tilde{x} \leq \|A\| \operatorname{diam} \tilde{x}.$$

ANOTHER ATTRIBUTE OF THE DIAMETER IS SLIGHTLY BELOW.

LET $\tau \to u_\tau$ BE A FUNCTION WITH VALUES $\mathfrak{x}$ AND THE EXISTING LEBESGUE INTEGRAL $\int_{\tau_0}^{\tau} u_\theta \, d\theta$. LET ALSO $\tau \to \tilde{u}_\tau$ BE A FUNCTION WITH VALUES $\tilde{\mathfrak{x}}$ AND THE NONEMPTY BOUNDED AUMANN INTEGRAL $\int_{\tau_0}^{\tau} \tilde{u}_\theta \, d\theta$, IMPLYING THAT

$$\operatorname{lev}_\lambda \int_{\tau_0}^{\tau} \tilde{u}_\theta \, d\theta = \left\{ \int_{\tau_0}^{\tau} u_\theta \, d\theta : \ u_\theta \in \operatorname{lev}_\lambda \tilde{u}_\theta \ \ \forall \theta \leq \tau \right\}$$

(RUNNING OVER ALL $u_\cdot$'S). LET $\tau \to v_\tau$ AND $\tau \to \tilde{w}_\tau$ BE ANOTHER PAIR OF SUCH FUNCTIONS. CLEARLY, IF THE FUNCTION $\tau \to \operatorname{diam} \tilde{u}_\tau$ IS LEBESGUE-INTEGRABLE THEN WE OBTAIN

$$\operatorname{diam} \int_{\tau_0}^{\tau} \tilde{u}_\theta \, d\theta \leq \int_{\tau_0}^{\tau} \operatorname{diam} \tilde{u}_\theta \, d\theta.$$

OBSERVE THAT FROM THE STANDPOINT OF THE AUMANN INTEGRAL AND DIAMETER THERE IS NO NEED TO DISTINGUISH SETS THEMSELVES AND THEIR CONVEX HULLS [4].

2. Description of the system

Consider a control system represented as

$$\frac{d}{d\tau} x = Ax + u_\tau ,$$

$$x\Big|_{\tau=0} = x_0 \in \tilde{x}_0 , \quad u_\tau \in \tilde{u}_\tau .$$

In what follows the known (symmetric) matrix $[A]$ is supposed to be stable (i. e. a matrix with negative eigenvalues). Thereby the matrices $[A]^{-1}$ and A^{-1} exist. We suppose also that

$$\tilde{u}_\tau = \{v_\tau\} + \tilde{w}_\tau \quad \forall \tau ,$$

with the function $\tilde{w}_.$ increasing (in the sense of inclusion) - in symbols:

$$\theta \leq \tau \Rightarrow \tilde{w}_\theta \subseteq \tilde{w}_\tau .$$

Then, clearly, the function $\tau \to \operatorname{diam} \tilde{u}_\tau$ is Lebesgue-integrable (owing to its monotonicity).

Given x_0 and $u_.$, the system's state is represented by the Cauchy formula

$$x_\tau = e^{A\tau} x_0 + \int_0^\tau e^{A(\tau-\theta)} u_\theta \, d\theta .$$

Given nothing but $\tilde{x}_0$ and $\tilde{u}_.$, the state is represented by

THE FUZZY ATTAINABLE SET

$$\tilde{x}_\tau = e^{A\tau}\tilde{x}_0 + \int_0^\tau e^{A(\tau-\theta)}\tilde{u}_\theta \, d\theta ,$$

AS CONSISTENT WITH ZADEH'S EXTENSION PRINCIPLE [3].

THROUGH THE DEFINITION OF AUMANN INTEGRAL THE PRECEDING FORMULA YIELDS

$$\tilde{x}_\tau = \tilde{x}_{\tau_0} + \int_{\tau_0}^{\tau} e^{A(\tau-\theta)} v_\theta \, d\theta + \int_{\tau_0}^{\tau} e^{A(\tau-\theta)} \tilde{w}_\theta \, d\theta \quad \forall \tau \geqslant \tau_0 .$$

THE FUNCTION $v_\cdot$ PLAYS THE ROLE OF PROGRAMMED CONTROL.

FROM THE WAŻEWSKI INEQUALITY [5], THAT GIVES AN UPPER (AND LOWER) BOUND FOR THE NORM OF THE SOLUTION OF A LINEAR CAUCHY PROBLEM, WE DERIVE

LEMMA 1.

$$\| e^A \| \leqslant \exp \frac{-1}{\| [A]^{-1} \|} .$$

THEREBY THE MATRIX $(E - e^A)^{-1}$ EXISTS, AND FROM ONE WELL-KNOWN INEQUALITY [6], THAT GIVES AN UPPER BOUND FOR THE NORM OF THE VON NEUMANN SERIES'S SUM, WE DERIVE

LEMMA 2.

$$\| (E - e^A)^{-1} \| \leqslant \left(1 - \exp \frac{-1}{\| [A]^{-1} \|} \right)^{-1} .$$

OBSERVE THAT $-1/\| [A]^{-1} \|$ IS THE MAXIMAL EIGENVALUE OF $[A]$.

3. Estimation of the uncertainty

If the uncertainty associated with a fuzzy set is to be measured by the diameter of this set, the uncertainty associated with $\tilde{x}_\tau$ can be estimated as follows.

Theorem 1.

$$\operatorname{diam} \tilde{x}_\tau \geqslant \left(1 - \exp \frac{\tau_0 - \tau}{\|[A]^{-1}\|}\right) \frac{\operatorname{diam} \tilde{w}_{\tau_0}}{\|A\|} .$$

Proof. We shall consider the case $\tau > \tau_0$. The definitions of Aumann integral and matrix integral yield

$$\int_{\tau_0}^{\tau} e^{A(\tau-\theta)} \tilde{w}_\theta \, d\theta \supseteq \int_{\tau_0}^{\tau} e^{A(\tau-\theta)} d\theta \, \tilde{w}_{\tau_0} = A^{-1}\left(E - e^{A(\tau-\tau_0)}\right) \tilde{w}_{\tau_0}.$$

Turning to the diameter's attributes, we obtain

$$\operatorname{diam} \tilde{x}_\tau = \operatorname{diam}\left(\tilde{x}_{\tau_0} + \int_{\tau_0}^{\tau} e^{A(\tau-\theta)} v_\theta \, d\theta + \int_{\tau_0}^{\tau} e^{A(\tau-\theta)} \tilde{w}_\theta \, d\theta\right) \geqslant$$

$$\geqslant \operatorname{diam} \int_{\tau_0}^{\tau} e^{A(\tau-\theta)} \tilde{w}_\theta \, d\theta \geqslant \frac{\operatorname{diam} \tilde{w}_{\tau_0}}{\|(E - e^{A(\tau-\tau_0)})^{-1}\| \, \|A\|} .$$

It remains to apply Lemma 2.

Thus the quantity

$$\frac{\operatorname{diam} \tilde{w}_{\tau_0}}{\|A\|}$$

estimates a progressive uncertainty concerning the system's state independently of $\tilde{x}_0$.

THEOREM 2.

$$\operatorname{diam} \tilde{x}_0 \geqslant \|[A]^{-1}\| \operatorname{diam} \tilde{w}_\tau \Rightarrow \operatorname{diam} \tilde{x}_\tau \leqslant \operatorname{diam} \tilde{x}_0 .$$

PROOF. LEMMA 1 YIELDS

$$\int_0^\tau \| e^{A(\tau-\theta)} \| d\theta \leqslant \left(1 - \| e^{A\tau} \| \right) \| [A]^{-1} \| .$$

TURNING TO THE DIAMETER'S ATTRIBUTES, WE OBTAIN

$$\operatorname{diam} \tilde{x}_\tau =$$

$$= \operatorname{diam}\left(e^{A\tau} \tilde{x}_0 + \int_0^\tau e^{A(\tau-\theta)} v_\theta d\theta + \int_0^\tau e^{A(\tau-\theta)} \tilde{w}_\theta d\theta \right) \leqslant$$

$$\leqslant \| e^{A\tau} \| \operatorname{diam} \tilde{x}_0 + \int_0^\tau \| e^{A(\tau-\theta)} \| d\theta \operatorname{diam} \tilde{w}_\tau \leqslant$$

$$\leqslant \| e^{A\tau} \| \operatorname{diam} \tilde{x}_0 + \left(1 - \| e^{A\tau} \| \right) \| [A]^{-1} \| \operatorname{diam} \tilde{w}_\tau .$$

IT REMAINS TO ASSUME THAT

$$\operatorname{diam} \tilde{x}_0 \geqslant \| [A]^{-1} \| \operatorname{diam} \tilde{w}_\tau .$$

THUS THE QUANTITY

$$\| [A]^{-1} \| \operatorname{diam} \tilde{w}_\tau$$

ESTIMATES A REGRESSIVE UNCERTAINTY CONCERNING THE SYSTEM'S STATE.

Conclusion

The above presented approach to mathematical description of uncertainty and of its evolution along the trajectories of dynamical systems operates in terms of fuzzy systems [7]. Naturally, there exist approaches in other terms, e. g. the information approach [8] where Renyi's entropy acts as diameter within a probabilistic framework.

References

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