FUZZY CONVEXITY, PERIPHERIC CORE AND α-NEAR NUCLEUS OF AN EXCHANGE ECONOMY

PART 1

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"Je ne puis que nommer les objets. Les signes les représentent. Je ne puis que parler des objets, je ne saurais les prononcer. Une proposition ne peut que dire d'une chose comment elle est, non ce qu'elle est.

Ludwig Wittgenstein
Tractatus Logico-philosophicus.

INTRODUCTION

In the framework of a research on the ordinal fuzzy games - different of those which were viewed by D. BUTNARIU (1979), (1980), where the payoffs functions are cardinal - two directions are possible. The first one, illustrated by the different works of J.P. AUBIN (1979)(1984), looks for deepening of the coalition concept - in allowing the set of coalitions to become a continuum.. For that purpose, we build an individual fuzzy membership to the coalition, but we do not influence the very foundation of the cooperation principle - the agent interest - i.e. his preference. In the case of the second approach, we tend to modelize a rational behavior in an imprecise world ; in other words, if the individual gambler perceives his preferences with imprecision, is it always pertinent to undertake the description of his rationality and have the games which are formed with gamblers who have non-booleian preferences, equilibrial solutions ? In the case of no-cooperation, the answer [BILLOT (1986)] is affirmative. Moreover, the set of Nash equilibria extends - and this under an hypothesis of no-local distinction - to the outside limits of the usual set. In a cooperative framework - that is the one adopted here - the solution is more difficult to exhibit. In fact, the theorems about the non-vacuity of the usual core [SCARF (1967), CORNWALL (1984),ROSENMULLER (1981)) are essentially based on the equilibration property of the game, property which needs, to be satisfied, the convexity of associated preorders to the payoffs functions of the gamblers. When the studied universe is economics, the property of convexity extends to the production set and to the consommation one. This hypothesis of convexity of preferences - if it is well interpretable - fixes nevertheless a constraint, a limit to the existence theorems of equilibria in a cooperative exchange economy. A lot of trials were elaborated just to go round this constraint and the one whose we are influenced by is tied to the extension of the imputations set of the core, in the neighbourhood of each of them (CORNWALL (1984), ROSENMULLER (1981)) .

Our approach joins around two concepts. Firstly, the fuzzy convexity - issued from a simple translation to the fuzzy sets of the very notion of convexity - means that a fuzzy subset is f-convex if all the linear combinations of two elements belonging to the subset, also belong to the subset, i.e. have a positive degree of membership, and that for a Zadeh set $\underline{M}$. On the other hand, the definition of a peripheric core and the one - induced - of the α-near nucleus allow to exhibit some theorems of non-vacuity which are less constraining even if the α-near nuclei convergence tends to the usual core which will be non-empty under usual hypothesis. This paper seeks to integrate these alterations on the framework of a purely economical application of a fuzzy game to a market game, just to study the influence of a cooperative - but imprecise - behavior on the equilibria of an economy which is, here, a pure exchange structure without production.

1 - FUZZY CONVEXITY

1.1 We consider an usual set $\underset{\sim}{X}$ of objects which are all capable of being classified : allocations, candidates, social states .... We also consider that this set is convex; it contains any linear combination of two of any of its elements.

1.2.1. We consider a fuzzy binary relation of indifference-preference, that we quote h.

1.2.2. The set of membership that we have chosen for the fuzzy binary relation, just like for the fuzzy subsets we have to exhibit, is the Zadeh-set or Łukasiewicz's one : $\underline{M} = (0,1)$.

1.2.3. The membership function of the fuzzy subset X, from $\underset{\sim}{X}$ ordered with h, is quoted μ_x and supposed to be continuous (A. BILLOT (1986))

1.2.4.1. We call preference degree from x to y : h(x.y).

1.2.4.2. We call preference degree from y to x : h(y.x).

1.2.4.3. If the preference degree from x to y is superior or equal to the preference degree from y to x, then we say that the membership degree of x to the fuzzy subset X is superior or equal to y's : $h(x.y) \geq h(y.x) \leftrightarrow \mu_x(x) \geq \mu_x(y)$.

1.2.4.4. h is reflexive : $\forall x \in \underline{X}$; $h(x.x) \geq o$

1.2.4.5. h is usual-transitive : $\forall (x.y.z) \in X^3$: if $h(x.y) \geq h(y.x)$ and $h(y.z) \geq h(z.y)$, then $h(x.z) \geq h(z.x)$.
i.e; if $\mu_x(x) \geq \mu_x(y)$ and $\mu_x(y) \geq \mu_x(z) \rightarrow \mu_x(x) \geq \mu_x(z)$.

1.2.4.6. The two propositions (1244) and (1245) define the fuzzy binary relation h like a preorder.

1.3.1. We call "proposition 3.1" : $\forall \alpha \in]0.1[\; ; \mu_X(\alpha x+(1-\alpha)y) = o$.
There is no linear combination of x and y which belongs to the fuzzy subset X.

1.3.2. We call "proposition 3.2" : $\exists \alpha \in]0.1[\; ; \mu_X(\alpha x+(1-\alpha)y) = o$.
There exists a linear combination of x and y which not belongs to the fuzzy subset X.

1.3.3. We call "proposition 3.3" : $\mu_X(x) > o$.
The element x belongs to X.

1.4.1. We call A the set of elements which belongs to $\underline{X}$ and verifies the proposition 3.1.

1.4.2. We call $\tilde{A}$ the set of elements which belongs to $\underline{X}$ and verifies the proposition 3.2.

1.4.3. We call $\underline{X}_o$ the set of elements which belongs to $\underline{X}$ and verifies the proposition 3.3.

1.5.1. It is obvious that the proposition (3.1) induces the proposition (3.2) and so : $\tilde{A} \subset A$.

1.5.2. It is obvious that the proposition ($\sim$3.2) is incompatible with the proposition (3.1), so it is clear that $\bar{\tilde{A}} \cap A = \emptyset$ with $\bar{\tilde{A}}$ the complementary set of $\tilde{A}$ in $\underline{X}$ and ($\sim$3.2) the negative proposition of (3.2).

<u>1.6.1.</u> The complementary set of $\underline{X}_o$ in $\underline{X}$, denoted $\bar{\underline{X}}_o$, is included in A.

In fact, if $\forall \alpha \in]0.1[$, $\mu_X(\alpha x+(1-\alpha)y) = o$,
it means with (123) that :
$\lim_{\alpha \to o^+} \mu_X(\alpha x+(1-\alpha)y) = o$. So, we can write : $\mu_X(x) = \mu_X(y) = o$.
The proposition (3.1) induces the negative proposition of (3.3), denoted ($\sim$3.3) : $\bar{\underline{X}}_o \subset A$.

1.6.2. The set $\tilde{A}$ is included in $\bar{\underline{X}}_o$.

In fact, if $\mu_X(x) = o$, it is obvious with (123) that $\lim_{\alpha \to -} \mu_X(\alpha x+(1-\alpha)y) = o$. So, there exists one $\alpha \varepsilon \,]0,1[$ such as $(\mu_X(x) = o)$ induces $\mu_X(\alpha x+(1-\alpha)y) = o$. The proposition (3.2) is induced by the negative proposition (∿3.3) : $\tilde{A} \subset \bar{\underline{X}}_o$.

1.6.3. The three following sets $\tilde{A}$, A and $\underline{X}_o$ are the same.

In fact, (162) tells us that $\tilde{A} \subset \bar{\underline{X}}_o$. On the other hand, (161) shows us that $\bar{\underline{X}}_o \subset A$.
We also know that (151) and (152) allow to write $A \supset \tilde{A}$ and $\bar{\tilde{A}} \cap A = \emptyset$. If $A \supset \tilde{A}$, it means that $\bar{\tilde{A}} \supset \bar{A}$: so, we can say that $\bar{\tilde{A}} \cap \tilde{A} \supset \bar{A} \cap \tilde{A}$. By definition, $\bar{\tilde{A}} \cap \tilde{A} = \emptyset$, so if $\emptyset \supset \bar{A} \cap \tilde{A}$, it means : $\bar{A} \cap \tilde{A} = \emptyset$. From that, we can induce that $\bar{A} \cup \tilde{A} = \underline{x}$ because (152). So, $\bar{A} \cup A = \bar{A} \cup \tilde{A}$. Because we have shown that $\bar{A} \cap \tilde{A} = \emptyset$ and $\bar{A} \cap A = \emptyset$ by definition, we can conclude that $A = \tilde{A} = \bar{\underline{X}}_o$.

1.6.4.1. A fuzzy subset X, issued from the convex set $\underline{x}$ ordered with h, defined in (1241), (1242) and (1243) is said f-convex, even when the complementary set of $\tilde{A}$ in $\underline{x}$ is equal to the very set $\bar{\underline{X}}_o$.

1.6.4.2. X is f-convex if, and only if $\{\forall \alpha \varepsilon [0.1] ; \mu_X(\alpha x+(1-\alpha)y) > o\}$ is equivalent to $(\mu_X(x) > o$ and $\mu_X(y) > o)$. In other words, even when two elements of $\underline{x}$ belongs to X, then any linear combination of these two elements also belongs to X with a membership $\mu_X(\alpha x+(1-\alpha)y)$.

1.7. Each preorder h defined with (1241), (1242) and (1243) generates a fuzzy subset X which is f-convex.

In fact, the proposition (163) means that the two sets $\tilde{A}$ and $\underline{x}_o$ are the same. So, their complementary sets are the same. $\bar{\tilde{A}} = \bar{\underline{X}}_o$; in that sense, we can establish the equivalence between the two following propositions : (∿3.2) and (∿3.3).

1.8. We point out that in the definition - propositions of a fuzzy preorder h, (1241), (1242) and (1243), there is no hypothesis in the convexity of concavity of the preorder h. The fuzzy convexity is so, totally independent of the eventual convexity of the preorder helping to define the fuzzy subset X.

2 - THE PERIPHERIC CORE

2.1. We consider the set of agents, the society, quoted S, formed with I agents.

2.1.1. We call any subset of S : a coalition.

2.1.2. Any coalition is, a priori, possible. Nevertheless, in order to carry away the agents to cooperate, it is necessary that a certain interest community brings them closer. This is the reason why the set of formed coalitions will be quoted CF and will be a part of P(S), where P(S) is the set of the parts - a priori possible coalitions. We call CF a structure of coalitions.

2.1.3.1. We allow the different agents who form the society S to belong to many coalitions of CF. Each coalition C which contains the agent i gets a certain part of representativity α_C^i.

2.1.3.2. So, any coalition belonging to CF is applied to a family of coefficients (α_C^i) for any agent i belonging to C, where α_C^i estimates the fraction of i which is represented by C.

2.1.3.3. On the other hand, each coalition C equally represents all its members. This value which estimates the representativity of the coalition C, is quoted α_C.$(\forall i j \varepsilon\ C\ ;\ \alpha_C^i = \alpha_C^j)$

2.1.4. A balanced structure of coalitions is a part CF of P(S) satisfying the following property. We can associate a positive number α_c, defined in (2133), to any coalition C belonging to the structure CF. Any agent must be totally represented with the coalitions in which he is contained ; that means that the sum of the α_c - where C is the coalition or coalitions which he belongs - is equal to 1, and that for any agent of S.

2.2.1. For any subset C of S, we appoint the set of vectors with components x_i, R^c, and that for an agent i belonging to the coalition C.

2.2.2.1. We can deduce from (221) : $R^S = R^I$

2.2.2.2. To any vector $\vec{a} = (a_1, a^2, — a_i, — a_I)$ of R^S, a vector $\pi_c \vec{a} = (a_i)_{i \varepsilon C}$ of R^c is associated which can be obtained in keeping it back to form $\pi_c \vec{a}$, the only components of $\vec{a}$ indexed by C.

2.2.2.3. π_c is a su■jective linear operator.

2.2.3. We call a cooperative game of I agents, the data, for any non-empty subset C of S, of a non-empty part $\Upsilon(c)$ of R^c satisfying the following property : $\Upsilon(c) - R^c_+ \subset \Upsilon(c)$.

2.2.4.1. The elements of S, specified in (211), will be called "gamblers".

2.2.4.2. The vectors of R^c will be called the C-imputations.

2.2.5.1. A coalition C of gamblers belonging to CF (212) totally blocks the imputation $\vec{x}$ if another imputation $\vec{y}$ exists, such as $\pi_c \vec{y} >> \pi_c \vec{x}$ (with $\vec{x}$ and $\vec{y}$ possible imputations).

2.2.5.2. We call a peripheric core $\underset{\sim}{CP}$ the fuzzy subset of the imputations which are either partially blocked or not blocked at all. i.e. $\underset{\sim}{CP} = \{\vec{x} \varepsilon R^S;\ \mu_{\underset{\sim}{CP}};\ \mu_{\underset{\sim}{CP}}(x) = 1$ if $\forall c \,\varepsilon\, CF$ (2251) is never satisfied and $0 \leq \mu_{\underset{\sim}{CP}}(x) < 1$ or else$\}$

2.2.5.3. The vector $\pi_C \vec{x}$ can be reviewed as the attribution of an utility level x_i to each member of the coalition C. Any gambler of C looks for maximizing his utility. The rule allowing the formation of cooperation between the gamblers corresponds to the fact that any agent i - in the different coalitions - will look for the one which guarantees him the best level of utility.

2.3.1. Let an exchange economy which is formed of ℓ goods.

2.3.2. The vector $\vec{E}$ of initial endowments represents all what the agents can demand. This economy is without production. The economic problem which is studied is the one of the partition of the initial endowments.

2.3.3 We call an allocation any vector $\vec{V} = (V_1, V_2, \text{---} V_i, \text{---} V_I)$ of $R^{\ell I}$. If this allocation satisfies : $\vec{E} = \sum_I V_i$, we call it a possible allocation.

2.3.4.1. The I gamblers-agents of S - are specified by their preferences, h_i, fuzzy preorders defined in (1241), (1242) and (1243).

2.3.4.2. The set of individual consommations V_i is a subset of R^{ℓ}.

2.3.5. Under the hypothesis of continuity of the fuzzy preorder, fuzzy connexity and compacity of V_i, there exists an individual utility function which is continuous (A.BILLOT (1986)).

2.3.6. The individual fuzzy utility function defined in (235) does not depend on the vector $\vec{V}$ of $R^{\ell I}$, but only on V_i. This hypothesis modelizes a selfish behaviour. This means that altruism and jealousy are not possible.

2.3.7. The initial endowments $\vec{E}$ have been initially allocated between the agents.

2.3.8. We call an economy of private property, the one which satisfies the two propositions (236) and (237).

2.4.1. In a economy of private property, satisfying the propositions (236) and (237), a coalition C belonging to CF (212) blocks the allocation $\vec{x}$ of $R^{\ell I}$ if complexes of goods, y^i of R^{ℓ} exist, such as any member of the coalition C prefers y_i to x_i and such as $\vec{y}$ is a possible allocation (233) : $C \in CF$; $\vec{x} \in R^{\ell I}$; $\vec{x}$ is blocked if $\pi_c \mu(y_i) >> \pi_c \mu(x_i)$ and $\vec{E} = \sum_I y_i$.

2.4.2.1. We call RB(C), the set of inside possible redistributions of C.

2.4.2.2 We call H(C), the set of utilities which the coalition C can simultaneously insure to its members.

2.4.2.3. Any agent i defines on RB($\mathcal{S}$) a fuzzy subset $\underset{\sim}{RB}_i(\mathcal{S})$ with his preorder h_i, according to the proposition (235).

2.4.3.1. We call $\underset{\sim}{CP}$ the peripheric-core of the economy, the fuzzy subset of the possible allocations which are either partially blocked or which are not blocked at all.

2.4.3.2. $\underset{\sim}{CP} = \{ \vec{x} \in RB(\mathcal{S})$: $\forall i \in C$, $\forall C \in CF$; $\mu_{\underset{\sim}{CP}}(\vec{x}) = 1$

if $\forall \vec{y} \in RB(C)$: $\mu_{\underset{\sim}{RB}_i(\mathcal{S})}(x_i) \geq \mu_{\underset{\sim}{RB}_i(\mathcal{S})}(y_i)$

and $\mu_{\underset{\sim}{CP}}(\vec{x}) = \underset{\substack{i \in C \\ C \in CF}}{\mathrm{Min}} \ \mu_{\underset{\sim}{RB}_i(\mathcal{S})}(x_i)$

if $\exists \ \vec{y} \in RB(C)$: $\mu_{\underset{\sim}{RB}_i(\mathcal{S})}(x_i) < \mu_{\underset{\sim}{RB}_i(\mathcal{S})}(y_i) \}$

2.4.3.3. We can remark that the peripheric core $\underset{\sim}{CP}$ is here defined as the intersection of the different $\underset{\sim}{RB}_i(\mathcal{S})$ which represent the fuzzy subset issued from the individual preorder which is applied to the set of socially possible allocations.

2.4.4. We call "associated market game" to the economy of private property, the cooperative game with I agents where (223) : $\forall C \in CF$; $\gamma(c) = H(c) - R^c_I$

2.4.5. We consider as current the proposition according to which, if the allocation $\vec{x}$ belongs to the peripheric core of the economy, then $\mu(\vec{x})$ belongs to the peripheric core of the market game.

2.4.6. We can easily deduce from (245) that, if the peripheric core of the market game is not empty, then the peripheric core of the economy is not empty either.

2.4.7.A For any system of agents' preferences in the economy, the peripheric core of the economy is non-empty. (We can see here that the usual restriction to the intra-muros core needs an additional hypothesis about the preorders in order that they insure the non-vacuity of the intra-muros core).

Our demonstration simply develops a parallel with the one which associates to the preorders the equilibration (balancedness) of the associated market game of the economy. We envisage a balanced structure of coalitions CF and $\vec{a} = (a_1, a_2, \text{——}, a_I)$ an imputation satisfying the following property : $\pi_c\ (\vec{a})$ belongs to $\gamma(c)$ defined in (223). The structure CF is balanced in the sense of (214) ; in other words, it is possible to associate any coalition C belonging to CF, with a positif coefficient α_c, in order that the sum of the α_c is equal to 1, and that for any agent of S. With that, we can write, knowing that C_i is the set of coalitions of CF which contain the agent i and $\vec{z} = (z_1, z_2, \text{——}, z_I)$ an allocation ; for any agent i belonging to the society S

$Z_i = \sum_{C \in C_i} \alpha_c\ y_i^c$ where y_i^c is defined in the couple of following properties : 1) y_c^i is a possible allocation according to (233) 2) $\forall i \in C\ ; u_i(y_i^c) \geq a_i$ (with u_i, the individual utility function issued from the preorder $\succsim_i$).

We can easily see that z_i is a convex combination of the y_i^c. Because R^ℓ is a convex set, it contains all the possible z_i. According to (161), (162), (163) and (17), any fuzzy subset defined in (1241), (1242), (1243) and (235) is f-convex in the sense of (1641) and (1642), and that for any nature of preorder, convex or not. So, the set of x_i, of R^ℓ, which satisfies $u_i(x_i) \geq a_i$, is f-convex. If we call X^{a_i} this fuzzy subset of allocations of R^ℓ (constrained by 236), then, if two of any allocations of R^ℓ belong to X^{a_i}, then any linear combination (convex) of them also belongs to X^{a_i}. So, we can establish this first result : $\forall i \in S$; $\mu_{X^{a_i}}(z_i) > o$.

[For $\underline{M} = [0.1]$ (122). It can be useful to "copy" this equation on one of the couple of properties satisfied by y_c^i which can be written : $\mu_{X^{a_i}}(y_c^i) > o$].

Now we just have to verify the possibility of $\vec{z}$. It is obvious. In fact, $\sum_{i \in S} z_i = \sum_{i \in S} \sum_{C \in C_i} \alpha_c \, y_c^i$ $= \sum_{C \in CF} \sum_{i \in C} \alpha_c \, y_c^i = \sum_{C \in CF} \alpha_c \left(\sum_{i \in C} y_c^i \right)$. We know that $\vec{y}_c$ is a possible allocation ; it means, according to (233) that $\sum_{i \in C} y_c^i = \sum_{i \in C} E_i$.

So, $\sum_{i \in S} z_i = \sum_{C \in CF} \sum_{i \in C} \alpha_c . E_i = \sum_{i \in S} E_i \left(\sum_{C \in C_i} \alpha_c \right) = \sum_{i \in S} E_i$.

Thus, we can write that $\vec{z}$ is a possible allocation in the sense of (233). So $\vec{z}$ belongs to RB(δ) according to (2421) ; in other words, $\vec{z}$ belongs to the set of the socially possible allocations. On the other hand, we have established that z_i belongs to R^ℓ, and that for any agent ; so $\vec{x}$ belongs to $H(c) - R_I^c$ that is to say to $\Upsilon(c)$. See proposition (244).

2.4.8. The proposition (247) determines the existence of a balanced solution in the framework of a game where the individual preorders are not necessary convex. Nevertheless, the peripheric core is a fuzzy subset. This means that the balanced solutions can belong with imprecision to the core of the economy. In other words, if no coalition totally belongs to the peripheric core of the economy (case of vacuity of the intra-muros core), then it is possible to envisage as "approximately balancing" some allocations whose membership degrees to the peripheric core is very close to the unity, but still inferior.

2.4.9.1. We call support of a fuzzy subset X, the usual set :
$\underline{X} = \{x \in X \; ; \mu_{\underline{X}}(x)=1 \text{ if } \mu_X(x)>o \text{ and } \mu_{\underline{X}}(x)=o \text{ if } \mu_X(x)=o\}$

2.4.9.2. We call α-cut of a fuzzy subset X, the usual set :
$\underline{X}_\alpha = \{x \in X \; ; \mu_{\underline{X}_\alpha}(x)=1 \text{ if } \mu_X(x)\geq\alpha \text{ and } \mu_{\underline{X}_\alpha}(x)=o \text{ if } \mu_X(x)<\alpha\}$

2.5.1. The support of the peripheric core $\underset{\sim}{CP}$ is an usual subset of RB($\mathcal{S}$).

2.5.2. The intra-muros core is the 1-cut of the peripheric core $\underset{\sim}{CP}$.

2.5.3. If the membership function of the peripheric core $\underset{\sim}{CP}$ is quasi-concave, then the intra-muros core is non-empty.

In fact, if the fuzzy preorders of the agents are convex, the membership function defined on $\underset{\sim}{RB}_i(\mathcal{S})$ (according to the propositions (1243) and (235))is quasi-concave. According to the proposition (2433) the peripheric core (in the valued part on $[0,1[$ (2432)) is defined as the intersection of the $\underset{\sim}{RB}_i(\mathcal{S})$. So, we know that an intersection of sets defined with a membership function which is quasi-concave is equally quasi-concave ; thus, we know that $\underset{\sim}{CP}$

is an "usually convex" set (definition of an epigraph). So, we can go back to the basis theorems (SCARF (1967)) according to which if the preorders of agents are convex, then the core is non-empty, because the proposition (252) defines the usual core (intra-muros core) as the 1-cut of the peripheric core.

2.5.4.1. It is possible to envisage some more precise theorems about the nature of the core of the economy. Thus, an usual theorem of the theory of cooperative games says that, under the couple of hypothesis of convexity and continuity of the preorders of the agents, the core of the economy is a closed, bounded and non-empty set of $R^{\ell I}$. If the propositions (1241), (1242), (1243), (1244),(1245) and (1246) are defined, no more on the Zadeh set (or Łukasiewicz's one) [0,1], but on the membership couple {0,1} - which corresponds, on one hand to the infirmation of the proposition (122), on the other hand to the return to the theory of the booleian sets - thus, the proposition (235) accompanied with a hypothesis of convexity of the preorders, insures us the compact non-vacuity of the core of the economy.

2.5.4.2. If we maintain as valid the proposition (122) - we are in the case where the membership degree is valued on [0,1] - we know that the two propositions (235) and (247A) are simultaneously coherent ; thus this means that the peripheric core is a compact fuzzy subset - it is its own closure which is non-empty.

The list of references is appended to Part 2 (to appear in BUSEFAL n°35 .