FUZZY PAN-INTEGRAL

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ABSTRACT

In this paper, the fuzzy pan-integral is introduced, and several elementary properties of the fuzzy pan-integrals are discussed. Furthermore, a transformation theorem of the fuzzy pan-integrals is proved, and it is also pointed out that the pan-integral and the fuzzy integral on a fuzzy set and the (N) fuzzy integral studied in [1,3,4] are all the special forms of the fuzzy pan-integrals. Finally, some convergence theorems of a sequence of the fuzzy pan-integrals are shown.

Keywords: Fuzzy measure, Fuzzy pan-space, Fuzzy pan-integral.

§1 BASIC CONCEPTS

The concepts given in this section (from Definition 1.1 to Definition 1.5) of this paper are introduced from [1,2].

Throughout this paper, let $R_+=[0,\infty)$, $\bar{R}_+=[0,\infty]$.

Definition 1.1 Let $\oplus$ be a two-place operation defined on $\bar{R}_+$, $(\bar{R}_+, \oplus)$ is called an ordered commutative semi-group with respect to $\oplus$, if the following conditions are satisfied:

(1) $a \oplus b = b \oplus a$;

(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;

(3) $a_1 \leq b_1,\ a_2 \leq b_2 \rightarrow a_1 \oplus a_2 \leq b_1 \oplus b_2$;

(4) $a \oplus 0=a$;

Where $a,b,c,a_i,b_i \in \overline{R}_+$, $(i=1,2)$, "0" is the number zero.

(5) If $\{a_n\}\subset\overline{R}_+$, $\{b_n\}\subset\overline{R}_+$, and $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ are existent, then $\lim_{n\to\infty}(a_n \oplus b_n)=\lim_{n\to\infty} a_n \oplus \lim_{n\to\infty} b_n$;

Hereafter, we denote $S_m=a_1 \oplus a_2 \oplus \cdots \oplus a_n=\bigoplus_{i=1}^{m} a_i$.

Definition 1.2 Let $\odot$ be another two-place operation defined on $(\overline{R}_+, \oplus)$, $(\overline{R}_+, \oplus, \odot)$ is said to be an ordered commutative semi-ring with respect to " $\oplus$ " and " $\odot$ ", if and only if

(6) $a\odot b=b\odot a$;

(7) $(a\odot b)\odot c=a\odot(b\odot c)$;

(8) $(a \oplus b)\odot c=(a\odot c) \oplus (b\odot c)$;

(9) $a_1\leq b_1$, $a_2\leq b_2 \longrightarrow a_1\odot a_2 \leq b_1\odot b_2$;

(10) $a\odot 0=0$;

(11) $a\neq 0$, $b\neq 0 \longrightarrow a\odot b\neq 0$;

(12) There exists a unit element $I\in\overline{R}_+$, such that

$a\odot I=I\odot a=a$;

Where $a,b,c,a_i,b_i\in\overline{R}_+$ $(i=1,2)$;"0" is the number zero.

(13) Whenever $\{a_n\}\subset\overline{R}_+$, $\{b_n\}\subset\overline{R}_+$, $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ are existent and finite, then $\lim_{n\to\infty}(a_n\odot b_n)=\lim_{n\to\infty} a_n\odot \lim_{n\to\infty} b_n$. Furthermore, if $a\in R_+$, and if $\lim_{n\to\infty} b_n$ is existent, then

$\lim_{n\to\infty}(a\odot b_n)=a\odot \lim_{n\to\infty} b_n$.

Example 1. $(\overline{R}_+,\vee,\wedge)$ is an ordered commutative semi-ring, where $\vee$ and $\wedge$ are maximum and minimum operations respectively. $I=\infty$ is the unit element of $(\overline{R}_+,\vee,\wedge)$.

Example 2. $(R_+, \vee, \cdot)$ is and ordered commutative semi-ring, where "$\cdot$" is common multiplication. $I=1$ is the unit element.

Definition 1.3 Let $\mathcal{F}(X)=\{\underset{\sim}{A}: \underset{\sim}{A}: X\to[0,1]\}$ be the set of all fuzzy subsets of X. The subset $\underset{\sim}{\mathcal{F}}$ of $\mathcal{F}(X)$ is called a fuzzy σ-algebra, if the following holds:

(1) $\phi, X\in\underset{\sim}{\mathcal{F}}$;

(2) If $\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$, then $\underset{\sim}{A}^{c}\in\underset{\sim}{\mathcal{F}}$;

(3) If $\{\underset{\sim}{A_n}\}\subset\underset{\sim}{\mathcal{F}}$, then $\bigcup_{n=1}^{\infty}\underset{\sim}{A_n}\in\underset{\sim}{\mathcal{F}}$.

Definition 1.4 A fuzzy measure is an extended real-valued set function $\underset{\sim}{\mu}:\underset{\sim}{\mathcal{F}}\to[0,\infty]$ with the properties:

(1) $\underset{\sim}{\mu}(\phi)=0$;

(2) For any $\underset{\sim}{A},\underset{\sim}{B}\in\underset{\sim}{\mathcal{F}}$, if $\underset{\sim}{A}\subset\underset{\sim}{B}$, then $\underset{\sim}{\mu}(\underset{\sim}{A})\leqslant\underset{\sim}{\mu}(\underset{\sim}{B})$;

(3) (Continuity from below) Whenever $\{\underset{\sim}{A_n}\}\subset\underset{\sim}{\mathcal{F}}$, $\underset{\sim}{A_n}\subset\underset{\sim}{A_{n+1}}$, $n=1,2,\cdots$, then $\underset{\sim}{\mu}(\bigcup_{n=1}^{\infty}\underset{\sim}{A_n})=\lim_{n\to\infty}\underset{\sim}{\mu}(\underset{\sim}{A_n})$;

(4) (Continuity from above) Whenever $\{\underset{\sim}{A_n}\}\subset\underset{\sim}{\mathcal{F}}$, $\underset{\sim}{A_n}\supset\underset{\sim}{A_{n+1}}$, $n=1,2,\cdots$, and there exists n_0 such that $\underset{\sim}{\mu}(\underset{\sim}{A_{n_0}})<\infty$, then $\underset{\sim}{\mu}(\bigcap_{n=1}^{\infty}\underset{\sim}{A_n})=\lim_{n\to\infty}\underset{\sim}{\mu}(\underset{\sim}{A_n})$.

Definition 1.5 If $\underset{\sim}{\mathcal{F}}$ is a fuzzy σ-algebra , and if $\underset{\sim}{\mu}$ is a fuzzy measure on $\underset{\sim}{\mathcal{F}}$, then we call $(X,\underset{\sim}{\mathcal{F}})$ a fuzzy measurable space and $(X,\underset{\sim}{\mathcal{F}},\underset{\sim}{\mu})$ a fuzzy measure space .

Definition 1.6 Let $(X,\underset{\sim}{\mathcal{F}},\underset{\sim}{\mu})$ be a fuzzy measure space and $(\overline{R}_+, \oplus, \odot)$ be an ordered commutative semi-ring, we call $(X, \underset{\sim}{\mathcal{F}}, \underset{\sim}{\mu}, \overline{R}_+, \oplus, \odot)$ a fuzzy pan-space .

§2 DEFINITION AND PROPERTIES OF FUZZY PAN-INTEGRAL

In this section, we will give the defimition of the fuzzy pan-integral and its elementary properties .

Definition 2.1 Let $(X,\underset{\sim}{\mathcal{F}})$ be a fuzzy measurable space, $(\bar{R}_+, \oplus, \odot)$ be an ordered commutative semi-ring, $\mathcal{B}=\{E;\ E\in\underset{\sim}{\mathcal{F}},\ E \text{ is a classical set}\}$ (It is clear that $\mathcal{B}$ is a classical σ-algebra contained in $\underset{\sim}{\mathcal{F}}$) . Then

(1) If I is the unit element of $(\bar{R}_+, \oplus, \odot)$, the extended real-valued function on X ;

$$\chi_E(x)=\begin{cases} I & \text{if } x\in E \\ 0 & \text{if } x\bar{\in} E \end{cases}$$

is called the characteristic function of E, where $E\in\mathcal{B}$;

(2) The real-valued function $s(x)$

$$s(x)=\bigoplus_{i=1}^{n}(\alpha_i\odot\chi_{E_i}(x))$$

is said to be a simple function on X, if $0\leqslant\alpha_i<\infty$, $E_i\in\mathcal{B}$, $i=1,2,\cdots n$, and if $\alpha_i=\alpha_j$, $E_i\cap E_j=\emptyset$, $i\neq j$; $\bigcup_{i=1}^{n}E_i=X$.

We denote the set of all simple functions on X by S .

It is easy to prove that a simple function given in Definition 2.1 has unique representation .

Definition 2.2 A mapping $f:\ X\to(-\infty,\infty)$ is called a measurable function on $\underset{\sim}{\mathcal{F}}$, if for any $\alpha\in[-\infty,\infty]$, we have

$F_\alpha=\{x;\ f(x)\geqslant\alpha\}\in\underset{\sim}{\mathcal{F}}$.

Denote the set of all measurable functions on $\underset{\sim}{\mathcal{F}}$ by $\underset{\sim}{M}$, and write $\underset{\sim}{M}^+=\{f;\ f\in\underset{\sim}{M},\ f\geqslant 0\}$.

Definition 2.3 Let $(X,\underset{\sim}{\mathcal{F}},\underset{\sim}{\mu},\bar{R}_+,\oplus,\odot)$ be a fuzzy pan-space, $\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$. For any $s(x)=\bigoplus_{i=1}^{n}(\alpha_i\odot\chi_{E_i}(x))\in S$, we write

$$P_{\underset{\sim}{A}}(s)=\bigoplus_{i=1}^{n}(\alpha_i\odot\underset{\sim}{\mu}(\underset{\sim}{A}\cap E_i)),$$

when $f\in\underset{\sim}{M}^+$, the fuzzy pan-integral of f on $\underset{\sim}{A}$ with respect to

$\underset{\sim}{\mu}$ is defined by

$$(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \triangleq \sup_{s\in s(f)} P_{\underset{\sim}{A}}(s)$$

where $s(f)=\{s:\ s\in S,\ 0\leq s\leq f\}$.

The fuzzy pan-integrals hold the following properties.

Proposition 2.1 Let $f,g\in \underset{\sim}{M}^+, \underset{\sim}{A},\underset{\sim}{B}\in \underset{\sim}{\mathcal{F}}$, then

(1) If $f\leq g$, then $(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$ for any $\underset{\sim}{A}\in\underset{\sim}{\mathcal{F}}$;

(2) $(p)\int_{\underset{\sim}{A}}(f\wedge g)d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \wedge (p)\int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$;

(3) $(p)\int_{\underset{\sim}{A}}(f\vee g)d\underset{\sim}{\mu} \geq (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \vee (p)\int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$;

(4) If $\underset{\sim}{A}\subset\underset{\sim}{B}$, then $(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$;

(5) $(p)\int_{\underset{\sim}{A}\cap\underset{\sim}{B}} f d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \wedge (p)\int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$;

(6) $(p)\int_{\underset{\sim}{A}\cup\underset{\sim}{B}} f d\underset{\sim}{\mu} \geq (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \vee (p)\int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$;

(7) If $\underset{\sim}{\mu}(\underset{\sim}{A})=0$, then $(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}=0$ for any $f\in\underset{\sim}{M}^+$;

(8) $(p)\int_{\underset{\sim}{A}} C d\underset{\sim}{\mu} \geq C\odot\underset{\sim}{\mu}(\underset{\sim}{A})$, $C\in R_+$.

Proof. (1) If $f\leq g$, then $s(f)\subset s(g)$, thus

$(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \sup_{s\in s(f)} P_{\underset{\sim}{A}}(s) \leq \sup_{s\in s(g)} P_{\underset{\sim}{A}}(s) = (p)\int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$.

(2) Since $f\wedge g\leq f$, $f\wedge g\leq g$, by (1), we have

$(p)\int_{\underset{\sim}{A}}(f\wedge g)d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$ and $(p)\int_{\underset{\sim}{A}}(f\wedge g)d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$, therefore, $(p)\int_{\underset{\sim}{A}}(f\wedge g)d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \wedge (p)\int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$.

(3) By using (1) , it is easy to prove this conclusion.

(4) If $\underset{\sim}{A}\subset\underset{\sim}{B}$, then $P_{\underset{\sim}{A}}(s)\leq P_{\underset{\sim}{B}}(s)$ and therefore

$(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$.

(5) It follows from (4) that $(p)\int_{\underset{\sim}{A}\cap\underset{\sim}{B}} f d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$ and $(p)\int_{\underset{\sim}{A}\cap\underset{\sim}{B}} f d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$, thus, we have

$(p)\int_{\underset{\sim}{A}\cap\underset{\sim}{B}} f d\underset{\sim}{\mu} \leq (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \wedge (p)\int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$.

(6) The proof of this conclusion is similar to (5) .

(7) When $\underset{\sim}{\mu}(\underset{\sim}{A})=0$, then $P_{\underset{\sim}{A}}(s)=0$ for any $s\in S$, thus $(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}=0$ for any $f\in \underset{\sim}{M}^+$.

(8) Observe that $s_0=C\odot\chi_X\in s(f)$, where $f\equiv C$. So, we have

$$(p)\int_{\underset{\sim}{A}} C d\underset{\sim}{\mu} \geq C\odot\underset{\sim}{\mu}(\underset{\sim}{A}).$$

Proposition 2.2 If $(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}=0$, then $\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_{\bar{0}})=0$, where $F_{\bar{0}}=\{x: f(x)>0\}$.

Proof. Suppose that $\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_{\bar{0}})=C>0$, since $\underset{\sim}{A}\cap F_{\frac{1}{n}}\nearrow \underset{\sim}{A}\cap F_{\bar{0}}$, by the continuity from below of $\underset{\sim}{\mu}$, we have $\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_{\frac{1}{n}})\to\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_{\bar{0}})=C$. thus there exists n_0, such that $\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_{\frac{1}{n_0}})\geq \frac{C}{2}>0$. Observe that $s_0=\frac{1}{n_0}\odot\chi_{F_{\frac{1}{n_0}}}\in s(f)$, it follows that

$$(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}=\sup_{s\in s(f)}\Big(\bigoplus_{i=1}^{n}(\alpha_i\odot\underset{\sim}{\mu}(\underset{\sim}{A}\cap E_i))\Big)\geq\frac{1}{n_0}\odot\underset{\sim}{\mu}(\underset{\sim}{A}\cap F_{\frac{1}{n_0}})>0 .$$

It is a contradiction.

§3 SOME PARTICULAR FORMS OF FUZZY PAN-INTEGRAL

Obviously, if $\underset{\sim}{A}$ is a classical set, then the fuzzy pan-integral on $\underset{\sim}{A}$ is the pan-integral defined in [1]. Furthermore, we have the following results:

Theorem 3.1 For any given $\underset{\sim}{A}\in\mathcal{F}$, if we define

$$\mu^*(E)\triangleq\underset{\sim}{\mu}(\underset{\sim}{A}\cap E) \text{ for any } E\in\mathcal{B},$$

then μ^* is a fuzzy measure on $\mathcal{B}$.

Proof. In fact, we have

(1) $\mu^*(\emptyset)=\underset{\sim}{\mu}(\underset{\sim}{A}\cap\emptyset)=\underset{\sim}{\mu}(\emptyset)=0$;

(2) If $E_1\subset E_2$, $E_1, E_2\in\mathcal{B}$, then $\mu^*(E_1)\leq\mu^*(E_2)$;

(3) Continuity from below : Let $\{E_n\}\subset\mathcal{B}$, $E_n\subset E_{n+1}$, $n=1,2,\cdots$, then $\mu^*(\bigcup_{n=1}^{\infty}E_n)=\underset{\sim}{\mu}(\underset{\sim}{A}\cap(\bigcup_{n=1}^{\infty}E_n))=\underset{\sim}{\mu}(\bigcup_{n=1}^{\infty}(\underset{\sim}{A}\cap E_n))=\lim_{n\to\infty}\underset{\sim}{\mu}(\underset{\sim}{A}\cap E_n)=\lim_{n\to\infty}\mu^*(E_n)$.

(4) Continuity from above : If $\{E_n\}\subset\mathcal{B}$, $E_n\supset E_{n+1}$, $n=1,2,\cdots$,

and if there exists n_0, such that $\mu^*(E_{n_0})<\infty$, then

$$\mu^*(\bigcap_{n=1}^{\infty} E_n)=\mu(A\cap(\bigcap_{n=1}^{\infty} E_n))=\mu(\bigcap_{n=1}^{\infty}(A\cap E_n))=\lim_{n\to\infty}\mu(A\cap E_n)=\lim_{n\to\infty}\mu^*(E_n) .$$

That is to say, μ^* is a fuzzy measure on $\mathscr{B}$.

Theerem 3.2 (Transformation theorem) Suppose that $(X, \mathscr{F}, \mu)$ is a fuzzy measure space, $A\in\mathscr{F}$, $f\in M^+$, we have

$$(p)\int_A f d\mu=(p)\int_X f d\mu^* .$$

Where $(p)\int_X f d\mu^*$ is the pan-integral defined in [1] and μ^* is the fuzzy measure defined in Theorem 3.1 .

Proof. $(p)\int_A f d\mu=\sup_{s\in s(f)}(\bigoplus_{i=1}^{n}(\alpha_i\odot\mu(A\cap E_i)))=\sup_{s\in s(f)}(\bigoplus_{i=1}^{n}(\alpha_i\odot\mu^*(E_i)))$

$=\sup_{s\in s(f)}(\bigoplus_{i=1}^{n}(\alpha_i\odot\mu^*(X\cap E_i)))=(p)\int_X f d\mu^*$.

The theorem is proved.

Using this transformation theorem, we can prove the following two particular forms of the fuzzy pan-integral .

Theorem 3.3 Let $(X,\mathscr{F}, \mu, \overline{R}_+, \oplus, \odot)$ be a fuzzy pan-space, $A\in\mathscr{F}$, $f\in M^+$. If $(\overline{R}_+, \oplus, \odot)=(\overline{R}_+, \vee, \wedge)$, then

$$(p)\int_A f d\mu=\int_A f d\mu$$

where $\int_A f d\mu=\sup_{\alpha\in[0,\infty]}(\alpha\wedge\mu(A\cap F_\alpha))$ is the fuzzy integral defined in [3].

Proof. For any $A\in\mathscr{F}$, $f\in M^+$, it follows from Theorem 4.2 in [1] and Theorem 2.2 in [3] and Theorem 3.2 , that

$$(p)\int_A f d\mu=(p)\int_X f d\mu^*=(s)\int_X f d\mu^*=\int_A f d\mu$$

where $(s)\int_X f d\mu^*$ is the Sugemo fuzzy integral on X with respect to μ^* .

Theorem 3.4 Let $(X,\mathscr{F}, \mu, \overline{R}_+, \oplus, \odot)$ be a fuzzy pan-space, $A\in\mathscr{F}$, $f\in M^+$. If $(\overline{R}_+, \oplus, \odot)=(\overline{R}_+, \vee, \cdot)$, then

$$(p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = (N)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$$

where $(N)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \sup_{\alpha>0}(\alpha \underset{\sim}{\mu}(\underset{\sim}{A} \cap F_\alpha))$ is the (N) fuzzy integral defined in [4].

Proof. Using Theorem 4.3 in [1] and Theorem 3.2, it is easy to prove this conclusion.

§ 4 CONVERGENCE THEOREMS

Theorem 4.1 Let $(X, \underset{\sim}{\mathscr{F}}, \underset{\sim}{\mu}, \overline{R}_+, \oplus, \odot)$ be a fuzzy pan-space, $\{f_n, f\} \subset \underset{\sim}{M}^+$, $f_n \leq f_{n+1}$, $n=1, 2, \cdots$, and if $f_n \to f$, then for any $\underset{\sim}{A} \in \underset{\sim}{\mathscr{F}}$, we have

$$\lim_{n\to\infty}(p)\int_{\underset{\sim}{A}} f_n d\underset{\sim}{\mu} = (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} .$$

Proof. By using Theorem 3.1 in [1] and Theorem 3.2, for any $\underset{\sim}{A} \in \underset{\sim}{\mathscr{F}}$, we have

$$\lim_{n\to\infty}(p)\int_{\underset{\sim}{A}} f_n d\underset{\sim}{\mu} = \lim_{n\to\infty}(p)\int_X f_n d\mu^* = (p)\int_X f d\mu^* = (p)\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} .$$

As the corollary of Theorem 4.1, we have the following

Theorem 4.2 (Fatou's Lemma) Let $(X, \underset{\sim}{\mathscr{F}}, \underset{\sim}{\mu}, \overline{R}_+, \oplus, \odot)$ be a fuzzy pan-space, if $\{f_n\} \subset \underset{\sim}{M}^+$, then

$$(p)\int_{\underset{\sim}{A}} \underline{\lim}_{n\to\infty} f_n d\underset{\sim}{\mu} \leq \underline{\lim}_{n\to\infty}(p)\int_{\underset{\sim}{A}} f_n d\underset{\sim}{\mu} .$$

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