SEMI-LATTICE STRUCTURE OF ALL EXTENSIONS OF POSSIBILITY MEASURE AND CONSONANT BELIEF FUNCTION ON THE FUZZY SET

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Abstract

The concepts of possibility measure and consonant belief function on the fuzzy set are introduced and the results analogous to [1] are obtained.

Keywords: Fuzzy set, Possibility measure,
Consonant belief function.

1. Introduction

In part 2 of this paper, we introduce the concepts: possibility measure and consonant belief function on the class of all fuzzy subsets of X, P-consistent and B-consistent of set function, and also discuss some of their properties. In part 3, we give two extension theorems of possibility measure and consonant belief function. In part 4, we discuss the semi-lattice structures of all extensions of possibility measure and consonant belief function.

Throughout this paper, let X be a set, F(X) be the class of all fuzzy subsets of X, and C be an arbitrary nonempty subset of F(X), μ be a mapping from C into the unit interval [0,1], and we make the following convention: $\bigcup_{\emptyset}\{\cdot\} = \emptyset$, $\bigcap_{\emptyset}\{\cdot\} = X$, $\sup_{\emptyset}\{\mu(\cdot)\} = 0$, $\inf_{\emptyset}\{\mu(\cdot)\} = 1$.

2. Possibility Measure and Consonant Belief Function on F(X)

Definition 2.1. A possibilty measure on F(X) is a non-negative real valued set function $\pi : F(X) \longrightarrow [0,1]$ with the property:

$$\pi(\bigcup_{t\in T} \underline{A}_t) = \sup_{t\in T} \pi(\underline{A}_t), \text{ whenever } \{\underline{A}_t : t \in T\} \subset F(X),$$

where T is an arbitrary index set.

Definition 2.2. A mapping $\underline{A}^*$ from X into the unit interval [0,1] is called a possibility distribution.

Theorem 2.1. A possibility distribution $\underline{A}^*$ defined on X can determine a possibility measure defined on F(X), and vice versa.

Proof. We define a mapping

$$\pi : F(X) \longrightarrow [0,1]$$
$$\underline{A} \longmapsto \sup_{x\in X}[(\underline{A} \cap \underline{A}^*)(x)],$$

then π is a possibility measure on F(X). In fact, for every $\{\underline{A}_t;\ t \in T\} \subset F(X)$, we have

$$(\bigcup_{t\in T} \underline{A}_t) = \sup_{x\in X}\{[(\bigcup_{t\in T} \underline{A}_t) \cap \underline{A}^*](x)\} = \sup_{x\in X}(\sup_{t\in T}(\underline{A}_t \cap \underline{A}^*)(x))$$

$$= \sup_{t\in T}(\sup_{x\in X}(\underline{A}_t \cap \underline{A}^*)(x)) = \sup_{t\in T} \pi(\underline{A}_t),$$

where T is an arbitrary index set.

On the other hand, let π be a possibility measure, then we define

$$\underline{A}^*(x) = \pi(A_x), \quad \forall\ x \in X,$$

where
$$A_x(y) = \begin{cases} 1, & \text{if } y = x \\ 0, & \text{if } y \neq x. \end{cases}$$

Obviously, $\underline{A}^*$ is a possibility distribution on X, and we complete the proof of the theorem.

Definition 2.3. A consonant belief function on F(X) is a nonnegative real-valued set function β : F(X) $\longrightarrow$ [0,1] with the property:

$$\beta(\bigcap_{t\in T}\underline{A}_t) = \inf_{t\in T}\beta(\underline{A}_t), \text{ whenever } \{\underline{A}_t : t \in T\} \subset F(X),$$

where T is an arbitrary index set.

Theorem 2.2. If β is a consonant belief function, then π defined by

$$\pi(\underline{A}) = 1 - \beta(\underline{A}^c),\ \underline{A} \in F(X),$$

is a possibility measure. Conversely, if π is a possibility measure, then β defined by

$$\beta(\underline{A}) = 1 - \pi(\underline{A}^c),\ \underline{A} \in F(X),$$

is a consonant belief function.

Definition 2.4. μ: C $\longrightarrow$ [0,1] is called P-consistent, if for every $\{\underline{A}_t ; t \in T\} \subset C$, $\underline{A} \in C$, with $\underline{A} \subset \bigcup_{t\in T}\underline{A}_t$, we have

$$\mu(\underline{A}) \leq \sup_{t\in T}\mu(\underline{A}_t).$$

Similarly, we introduce

Definition 2.5. μ: C $\longrightarrow$ [0,1] is called B-consistent, if for every $\{\underline{A}_t :\ t \in T\} \subset C$, $\underline{A} \in C$, with $\underline{A} \supset \bigcap_{t\in T} \underline{A}_t$, we have

$$\mu(\underline{A}) \geq \inf_{t\in T} \mu(\underline{A}_t).$$

3. Extension Theorems

Theorem 3.1. μ can be extended to a possibility measure on $F(X)$, if and only if μ is P-consistent.

Proof. Necessity. Obvious.

Sufficiency. If we define

$$\pi : F(X) \longrightarrow [0,1]$$

$$\underline{B} \longmapsto \sup_{x\in X} \inf_{\substack{(\bigcup_{s\in S^*} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C}} \sup_{s\in S^*} \mu(\underline{E}_s), \qquad (*)$$

where S^* is arbitrary index set, then π is a possibility measure on $F(X)$, and a extension of μ on C. To conclude the assertions, we first prove that π is a possibility measure. In fact, the monotonicity of μ is obvious. By the monotonicity of π, we have, for every $\{\underline{A}_t;\ t \in T\} \subset F(X)$,

$$\pi(\underline{A}_t) \leq \pi(\bigcup_{t\in T} \underline{A}_t),$$

and hence

$$\sup_{t\in T} \pi(\underline{A}_t) \leq \pi(\bigcup_{t\in T} \underline{A}_t),$$

where T is an arbitrary index set.

On the other hand, for every $x \in X$, $t \in T$, and for any $\varepsilon > 0$, there exists $\{\underline{E}_s;\ s \in S_t^*\} \subset C$ such that

$$(\bigcup_{s\in S_t^*} \underline{E}_s)(x) \geq \underline{A}_t(x),$$

and

$$\inf_{\substack{(\bigcup_{s\in S_t^x} \underline{E}_s)(x) \geq \underline{A}_t(x) \\ \underline{E}_s \in C}} \sup_{s\in S_t^x} \mu(\underline{E}_s) \geq \sup_{s\in S_t^x} \mu(\underline{E}_s) - \varepsilon .$$

Since

$$(\bigcup_{t\in T} \bigcup_{s\in S_t^x} \underline{E}_s)(x) \geq \bigvee_{t\in T} \underline{A}_t(x) = (\bigcup_{t\in T} \underline{A}_t)(x),$$

it follows that

$$\sup_{t\in T} \inf_{\substack{(\bigcup_{s\in S_t^x} \underline{E}_s)(x) \geq \underline{A}_t(x) \\ \underline{E}_s \in C}} \sup_{s\in S_t^x} \mu(\underline{E}_s) \geq \sup_{t\in T} \sup_{s\in S_t^x} \mu(\underline{E}_s) - \varepsilon$$

$$= \sup_{s\in \bigcup_{t\in T} S_t^x} \mu(\underline{E}_s) - \varepsilon \geq \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq (\bigcup_{t\in T} \underline{A}_t)(x) \\ \underline{E}_s \in C}} \sup_{s\in S^x} \mu(\underline{E}_s) - \varepsilon,$$

this shows that

$$\sup_{x\in X} \sup_{t\in T} \inf_{\substack{(\bigcup_{s\in S_t^x} \underline{E}_s)(x) \geq \underline{A}_t(x) \\ \underline{E}_s \in C}} \sup_{s\in S_t^x} \mu(\underline{E}_s)$$

$$\geq \sup_{x\in X} \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq (\bigcup_{t\in T} \underline{A}_t)(x) \\ \underline{E}_s \in C}} \sup_{s\in S^x} \mu(\underline{E}_s) - \varepsilon ,$$

it yields that

$$\sup_{t\in T} \pi(\underline{A}_t) \geq \pi(\bigcup_{t\in T} \underline{A}_t).$$

Consequently,

$$\sup_{t\in T} \pi(\underline{A}_t) = \pi(\bigcup_{t\in T} \underline{A}_t),$$

which means π is a possibility measure.

Next, we prove that π is an extension of μ on C. In fact, for every $\underline{B} \in C$, we have

$$\pi(\underline{B}) = \sup_{x\in X} \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C}} \sup_{s\in S^x} \mu(\underline{E}_s) \leq \sup_{x\in X} \mu(\underline{B}) = \mu(\underline{B}).$$

On the other hand, for any $\varepsilon > 0$, every $x \in X$ there exists $\{\underline{E}_s;\ s \in S^x\} \subset C$ such that

$$\underline{B}(x) \leq (\bigcup_{s\in S^x} \underline{E}_s)(x) \leq (\bigcup_{s\in \bigcup_{x\in X} S^x} \underline{E}_s)(x),$$

and

$$\inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C}} \sup_{s\in S^x} \mu(\underline{E}\ ) \geq \sup_{s\in S^x} \mu(\underline{E}_s) - \varepsilon,$$

hence, by using the P-consistence of μ,

$$\pi(\underline{B}) = \sup_{x\in X} \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C}} \sup_{s\in S^x} \mu(\underline{E}_s)$$

$$\geq \sup_{x\in X} \sup_{s\in S^x} \mu(\underline{E}_s) - \varepsilon = \sup_{s\in \bigcup_{x\in X} S^x} \mu(\underline{E}_s) - \varepsilon \geq \mu(\underline{B}) - \varepsilon;$$

therefore

$$\pi(\underline{B}) \geq \mu(\underline{B}).$$

Consequently,

$$\pi(\underline{B}) = \mu(\underline{B}),$$

and we complete the proof of the theorem. ∎

Theorem 3.2. μ can be extended to a consonant belief function on $F(X)$, if and only if μ is B-consistent.

Proof. The proof is similar to that of theorem 3.1, we only note that if μ is B-consistent, then β is a consonant belief function on $F(X)$, and it is an extension of μ, where β is defined by

$$\beta : F(X) \longrightarrow [0,1]$$

$$\underline{A} \longmapsto \inf_{x\in X} \sup_{\substack{(\bigcap_{s\in S^x} \underline{E}_s)(x) \le \underline{A}(x) \\ \underline{E}_s \in C}} \inf_{s\in S^x} \mu(\underline{E}_s). \qquad (**)$$

4. Semi-lattice Structure of All Extensions

In usual case, the extension of a mapping u with P-consistent from an arbitrary nonempty class of the fuzzy subsets of X into the unit interval [0,1] to a possibility measure on F(X) may not be unique. Similarly, the extension of a mapping μ with B-consistent from an arbitrary nonempty class of the fuzzy subsets of X into the unit interval [0,1] to a consonant belief function on F(X) may not be unique either. All extensions of possibility measure (consonant belief function) is denoted $E_\pi(\mu)$ ($E_\beta(\mu)$). By using theorem 3.1 (3.2), we know that $E_\pi(\mu)$ ($E_\beta(\mu)$) is nonempty, if μ is P-consistent (B-consistent).

For two mappings $\mu_1 : F(X) \longrightarrow [0,1]$ and $\mu_2 : F(X) \longrightarrow [0,1]$, we define ordering relation " $\le$ " :

$$\mu_1 \le \mu_2 \text{ if and only if } \mu_1(\underline{A}) \le \mu_2(\underline{A}), \forall\, \underline{A} \in F(X).$$

It is easy to prove that " $\le$ " is a partial ordering relation on $E_\pi(\mu)$ (Similarly, on $E_\beta(\mu)$). Therefore the least upper bound of μ_1, $\mu_2 \in E_\pi(\mu)$ can be defined by

$$(\sup\{\mu_1, \mu_2\})(\underline{A}) = \mu_1(\underline{A}) \vee \mu_2(\underline{A}), \text{ for all } \underline{A} \in F(X).$$

Similarly, the greatest lower bound can be defined by

$$(\inf\{\mu_1, \mu_2\})(\underline{A}) = \mu_1(\underline{A}) \wedge \mu_2(\underline{A}), \text{ for all } \underline{A} \in F(X).$$

Theorem 4.1. ($E_\pi(\mu)$, $\le$) is an upper semi-lattice, and the extension π defined by (*) is the greatest element of $E_\pi(\mu)$.

Proof. a) Obviously, $(E_\pi(\mu), \leq)$ is an upper semi-lattice.

b) The extension π defined by (*) is the greatest element of the $E_\pi(\mu)$. For arbitrary $\pi' \in E_\pi(u)$, $\underline{B} \in F(X)$, we define

$$\underline{F}_x(y) = \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C}} (\bigcup_{s\in S^x} \underline{E}_s)(x) \quad \text{if } y = x;$$

$$= 0 \qquad\qquad \text{if } y \neq x,$$

for every $x \in X$. If $\underline{B}(x) \leq (\bigcup_{s\in S^x} \underline{E})(x)$, $\underline{E}_s \in C$, we have

$$\underline{F}_x \subset \bigcup_{s\in S^x} \underline{E}_s,$$

hence

$$\pi'(\underline{F}_x) \leq \sup_{s\in S^x} \pi'(\underline{E}_s),$$

therefore

$$\pi'(\underline{F}_x) \leq \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C}} \sup_{s\in S^x} \pi'(\underline{E}_s)$$

$$= \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C}} \sup_{s\in S^x} \mu(\underline{E}_s),$$

for every $x \in X$, it follows, by using

$$\underline{B}(x) \leq \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C}} (\bigcup_{s\in S^x} \underline{E}_s)(x) = \underline{F}_x(x) \leq (\bigcup_{x\in X} \underline{F}_x)(x),$$

that

$$\pi(\underline{B}) = \sup_{x\in X} \inf_{\substack{(\bigcup_{s\in S^x} \underline{E}_s)(x) \geq \underline{B}(x) \\ \underline{E}_s \in C}} \sup_{s\in S^x} \mu(\underline{E}_s) \geq \sup_{x\in X} \pi'(\underline{F}_x)$$

$$= \pi'(\bigcup_{x\in X} \underline{F}_x) \geq \pi'(\underline{B}).$$

Similarly, we can prove

Theorem 4.2. $(E_\beta(\mu), \leq)$ is a lower semi-lattice, and the extension β defined by (**) is the least element of $E_\beta(\mu)$.

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References

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