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SOME NEW RESULTS FOR P-MEASURE

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1. EQUIVALENT DEFINITIONS OF P-MEASURE

By fuzzy measurable space we mean a pair (Ω, σ), where Ω is a crisp set of elementary events and σ is a soft fuzzy σ-algebra i.e. fuzzy algebra in the sense of Khalili [1] notcontaining $\left[\frac{1}{2}\right]_{\Omega}: \Omega \to \left\{\frac{1}{2}\right\}$ (see [4]). Elements of σ are interpreted as fuzzy random events. In [4] probability of fuzzy random event is described by fuzzy P-measure which is compatible with following definitions ($\mathbb{F}(\Omega)$ is the family of all fuzzy subsets in Ω).

Definition 1.1. [3] : An element $\mu \in \mathbb{F}(\Omega)$ will be called a W-universum if it contains its complement (i.e. $\mu \geqslant 1-\mu$).

Definition 1.2. [3] : An element $\mu \in \mathbb{F}(\Omega)$ will be called a W-empty set if it is contained in its complement (i.e. $\mu \leqslant 1-\mu$).

Definition 1.3. [3] : Elements $\mu, \nu \in \mathbb{F}(\Omega)$ will be called W-separated fuzzy subsets if the first is contained in the complement of the second (i.e. $\mu \leqslant 1-\nu$).

The following definition of fuzzy P-measure was given in [4].

Definition 1.4. [4] : A mapping $p: \mathfrak{S} \rightarrow \mathbb{R}^+ \cup \{0\}$ with the properties

(P1) if $\mu \in \mathfrak{S}$ is a W-universum then

$$p(\mu) = 1 \quad ;$$

(P2) for any sequence $\{\mu_n\} \subset \mathfrak{S}$ of pairwise W-separated fuzzy subsets we have

$$p\left(\sup_n \{\mu_n\}\right) = \sum_n p(\mu_n) \tag{1.1}$$

will be called a fuzzy P-measure.

Moreover, we have following equivalent definitions of fuzzy P-measure.

Theorem 1.1.: A mapping $p: \mathfrak{S} \rightarrow \mathbb{R}^+ \cup \{0\}$ is a fuzzy P-measure iff it satisfies:

(P3) for each such sequence $\{\mu_n\} \subset \mathfrak{S}$ of pairwise W-separated fuzzy subsets that $\sup_n \{\mu_n\}$ is a W-universum we have

$$1 = \sum_n p(\mu_n) \quad . \tag{1.2}$$

Proof: The necessity is proved in [4] . On the other side, the condition (P1) follows immediatelly from (P3). Furthermore, the condition (P3) implies.

$$p(1 - \mu) = 1 - p(\mu) \tag{1.3}$$

for each $\mu \in \mathfrak{S}$. Therefore, for any $\{\mu_n\} \subset \mathfrak{S}$ of W-separated fuzzy subsets we have

$$p\left(\sup_k \{\mu_k\} \vee \left(1 - \sup_n \{\mu_n\}\right)\right) = 1 = \sum_k p(\mu_k) + p\left(1 - \sup_n \{\mu_n\}\right) =$$

$$= \sum_k p(\mu_k) + 1 - p\left(\sup_n \{\mu_n\}\right)$$

because:

- μ_k and $1 - \sup_n \{\mu_n\}$ are W-separated for each positive integer k ;
- $\sup_k \{\mu_k\} \vee (1 - \sup_n \{\mu_n\})$ is a W-universum (see [3]).

So, the condition (P2) holds. ■

Theorem 1.2.: A mapping $p: \mathfrak{S} \to \mathbb{R}^+ \cup \{0\}$ is a fuzzy P-measure iff it satisfies (P1) and:

(P4) if $\mu, \nu \in \mathfrak{S}$ are W-separated then

$$p(\mu \vee \nu) = p(\mu) + p(\nu) \quad ; \tag{1.4}$$

(P5) if $\{\mu_n\} \subset \mathfrak{S}$ monotonously tends from below to $\mu \in \mathfrak{S}$ then $\{p(\mu_n)\}$ tends from below to $p(\mu)$.

Proof: The necessity is proved in [4] . If $\{\mu_n\} \subset \mathfrak{S}$ is any sequence of W-separated fuzzy subsets then, using the mathematical induction and (P4), we get

$$p(\max_{n \leq k} \{\mu_n\}) = \sum_{n=1}^{k} p(\mu_n)$$

for each positive integer k . Thus, by means of (P5), we obtain (P2). ■

Observe that the definition of W-separateness given above (Def.1.3) is not only one possible. Perhaps the more natural would be the following definitions.

Definition 1.5. [7] : Elements $\mu, \nu \in \mathbb{F}(\Omega)$ will be called F-separated if their intersection is a W-empty set (i.e. $\mu \wedge \nu \leq [\frac{1}{2}]_\Omega$).

Of course W-separated fuzzy subsets are F-separated but not vice versa. If we change in the Definition 1.4. the notion of W-separateness by the notion of F-separateness, we obtain formally stronger version of the notion of fuzzy P-measure. Nevertheless, we have:

Theorem 1.3. [7] : A mapping $p: \mathfrak{S} \to \mathbb{R}^+ \cup \{0\}$ is a fuzzy P-measure iff it satisfies (P1) and:

(P6) for any sequence $\{\mu_n\} \subset \mathfrak{S}$ of pairwise F-separated fuzzy subsets we have (1.1).

Theorem 1.4.: A mapping $p: \mathfrak{S} \to \mathbb{R}^+ \cup \{0\}$ is a fuzzy P-measure iff it satisfies:

(P7) for each such sequence $\{\mu_n\} \subset \mathfrak{S}$ of pairwise F-separated fuzzy subsets that $\sup_n \{\mu_n\}$ is a W-universum, we have (1.2).

Proof: The necessity is self-evident. Since μ and $1-\mu$ are F-separated for any $\mu \in \mathfrak{S}$, (P7) implies (1.3). Thus, in like manner, as (P2) in the proof of the Theorem 1.1. we get (P6). ■

Theorem 1.5.: A mapping $p: \mathfrak{S} \to \mathbb{R}^+ \cup \{0\}$ is a fuzzy P-measure iff it satisfies (P1), (P5) and:

(P8) if $\mu, \nu \in \mathfrak{S}$ are F-separated then (1.4).

The last theorem can be proved by analogous way as the Theorem 1.2.

2. AN EXTENDED DOMAIN OF THE BAYES FORMULA

Take into account the following families of fuzzy subsets

$$\mathbb{K}(\mathfrak{S}) = \{A: A \in 2^{\Omega}, \exists\, \mu \in \mathfrak{S} \;;\; K(\mu) \subset A \subset L(\mu)\},$$

$$\mathbb{E}(\mathfrak{S}) = \{\mu ; \mu \in \mathbb{F}(\Omega), \exists\, A, B \in \mathbb{K}(\mathfrak{S}) \;;\; A \subset B,\ A = K(\mu)\ \&\ B = L(\mu)\}$$

$$\mathbb{E}^*(\mathfrak{S}) = \{\mu: \mu \in \mathbb{E}(\mathfrak{S}), \exists\, \nu \in \mathfrak{S} \;;\; \mu \wedge (1-\mu) \leqslant \nu \wedge (1-\nu)\},$$

$$\widetilde{\mathbb{E}}(\sigma) = \{\mu : \mu \in \mathbb{E}(\sigma), \exists \nu \in \sigma \; ; \; K^*(\mu) \subset K^*(\nu)\}$$

where

$$K(\mu) = \{\omega : \omega \in \Omega, \; \mu(\omega) > \tfrac{1}{2}\} \; ,$$
$$K^*(\mu) = \{\omega : \omega \in \Omega, \; \mu(\omega) = \tfrac{1}{2}\} \; ,$$
$$L(\mu) = K(\mu) \cup K^*(\mu)$$

for each $\mu \in \mathbb{F}(\Omega)$. Then we have $\mathbb{E}^*(\sigma) \subset \widetilde{\mathbb{E}}(\sigma) \subset \mathbb{E}(\sigma)$ and

Theorem 2.1. [5] : $\mathbb{K}(\sigma)$ is a crisp σ-algebra.

Theorem 2.2. [5] : $\mathbb{E}^*(\sigma)$ is a soft fuzzy σ-algebra.

Theorem 2.3. [6] : $\widetilde{\mathbb{E}}(\sigma)$ is a soft fuzzy algebra.

Theorem 2.4. [5] : $\mathbb{E}(\sigma)$ is a fuzzy algebra containing $[\![\tfrac{1}{2}]\!]_{\Omega}$.

The problem of the smallest soft fuzzy -algebra containing $\widetilde{\mathbb{E}}(\sigma)$ is open.

Let $p : \sigma \to \mathbb{R}^+ \cup \{0\}$ be a fixed P-measure which is interpreted as probability of fuzzy events. Recall to mind that Antoniewicz and Ostasiewicz propose to describe probability of fuzzy events by means of the plausibility measure defined as follows:

Definition 2.1. [2] : A mapping $\mathrm{pls} : \sigma \to [0,1]$ is a plausibility measure if it satisfies

$$\mathrm{pls}(0_{\Omega}) = 0 \; ; \tag{2.1}$$
$$\mathrm{pls}(1_{\Omega}) = 1 \; ; \tag{2.2}$$
$$\mu \leqslant \nu \Rightarrow \mathrm{pls}(\mu) \leqslant \mathrm{pls}(\nu) \; ; \tag{2.3}$$
$$\mathrm{pls}(\mu \vee \nu) = \mathrm{pls}(\mu) + \mathrm{pls}(\nu) - \mathrm{pls}(\mu \wedge \nu) \tag{2.4}$$

for each $\mu, \nu \in \sigma$.

Any fuzzy P-measure is a plausibility measure (see [4]). Furthermore, we have:

Theorem 2.5: Let $\nu \in \mathfrak{S}$. Then, for any plausibility measure, we have:

- $\mathrm{pls}(\mu \vee \nu) = \mathrm{pls}(\mu)$ for all $\mu \in \mathfrak{S}$ iff $\mathrm{pls}(\nu) = 0$;
- $\mathrm{pls}(\mu \wedge \nu) = \mathrm{pls}(\mu)$ for all $\mu \in \mathfrak{S}$ iff $\mathrm{pls}(\nu) = 1$.

Proof: If for all $\mu \in \mathfrak{S}$ we have $\mathrm{pls}(\mu \vee \nu) = \mathrm{pls}(\mu)$ then, for $\mu = \mathbb{0}_{\Omega}$, from (2.1), we obtain

$$0 = \mathrm{pls}(\mathbb{0}_{\Omega}) = \mathrm{pls}(\mathbb{0}_{\Omega} \vee \nu) = \mathrm{pls}(\mu).$$

If $\mathrm{pls}(\nu) = 0$ then for all $\mu \in \mathfrak{S}$ we can write

$$\mathrm{pls}(\mu \vee \nu) = \mathrm{pls}(\mu) + \mathrm{pls}(\nu) - \mathrm{pls}(\mu \wedge \nu) = \mathrm{pls}(\mu)$$

because $0 \leqslant \mathrm{pls}(\mu \wedge \nu) \leqslant \mathrm{pls}(\nu) = 0$.

If, for all $\mu \in \mathfrak{S}$, we have $\mathrm{pls}(\mu \wedge \nu) = \mathrm{pls}(\mu)$ then, for $\mu = \mathbb{1}_{\Omega}$, by (2.2) we get

$$1 = \mathrm{pls}(\mathbb{1}_{\Omega}) = \mathrm{pls}(\mathbb{1}_{\Omega} \wedge \nu) = \mathrm{pls}(\nu).$$

If $\mathrm{pls}(\nu) = 1$ then, for all $\mu \in \mathfrak{S}$, we have $1 \geqslant \mathrm{pls}(\mu \vee \nu) \geqslant \mathrm{pls}(\nu) = 1$ and thus, using (2.4), we obtain

$$\mathrm{pls}(\mu \wedge \nu) = \mathrm{pls}(\mu) + \mathrm{pls}(\nu) - \mathrm{pls}(\mu \vee \nu) = \mathrm{pls}(\mu). \blacksquare$$

The further our considerations are based on the following theorems.

Theorem 2.6. [5]: The mapping $P: \mathbb{K}(\mathfrak{S}) \to [0,1]$, given by

$$\forall A \in \mathbb{K}(\mathfrak{S}) \qquad P(A) = p(\mu) \quad \text{if} \quad K(\mu) \subset A \subset L(\mu) \tag{2.5}$$

is well-defined usual probability measure on $\mathbb{K}(\mathfrak{S})$.

Theorem 2.7. [5]: The mapping $\bar{p}: \tilde{\mathbb{E}}(\mathfrak{S}) \to [0,1]$, defined by (2.5) and

$$\hat{p}(\mu) = P(K(\mu)) \tag{2.6}$$

for each $\mu \in \widetilde{\mathbb{E}}(\sigma)$, is the unique extension of fuzzy P-measure p on σ which is a fuzzy P-measure on $\widetilde{\mathbb{E}}(\sigma)$.

Theorem 2.8. [5] : The mapping $\bar{p}: \mathbb{E}(\sigma) \to [0,1]$, defined by (2.5) and (2.6) for each $\mu \in \mathbb{E}(\sigma)$, is the unique monotonic extension of $\bar{p}$ on $\mathbb{E}^*(\sigma)$ to $\mathbb{E}(\sigma)$ which satisfies (P6).

Theorem 2.9. [6] : The mapping $\hat{p}: \widetilde{\mathbb{E}}(\sigma) \to [0,1]$, defined by (2.5) and

$$\hat{p}(\mu) = P(L(\mu)) \tag{2.7}$$

for each $\mu \in \widetilde{\mathbb{E}}(\sigma)$, is the unique extension of p on σ to $\widetilde{\mathbb{E}}(\sigma)$ which is a fuzzy P-measure on $\widetilde{\mathbb{E}}(\sigma)$.

Theorem 2.10 [6] : If a plausibility measure $pls: \mathbb{E}(\sigma) \to [0,1]$ satisfies

$$\forall \mu \in \mathbb{E}^*(\sigma) \qquad pls(\mu) = \bar{p}(\mu) = \hat{p}(\mu) \tag{2.8}$$

then it fulfils

$$\forall \mu \in \mathbb{E}(\sigma) \qquad \bar{p}(\mu) \leqslant pls(\mu) \leqslant \hat{p}(\mu) . \tag{2.9}$$

So, the mappings $\bar{p}$ and $\hat{p}$ can be interpreted respectively as a lower and an upper extensions of fuzzy P-measure on σ to $\mathbb{E}(\sigma)$. Each plausibility measure satisfying (2.8) will be called an extension of fuzzy P-measure p on σ to $\mathbb{E}(\sigma)$.

Let $pls^*: \mathbb{E}(\sigma) \to [0,1]$ be a fixed extension of fuzzy P-measure p on σ to $\mathbb{E}(\sigma)$. Then, for each possible fuzzy condition $\nu \in \mathbb{E}(\sigma)$ (i.e. $pls^*(\nu) \neq 0$), a conditional plausibility $pls^*(\cdot|\nu): \mathbb{E}(\sigma) \to [0,1]$ is given by

$$\mathrm{pls}^*(\mu|\nu) = \frac{\mathrm{pls}^*(\mu\wedge\nu)}{\mathrm{pls}^*(\nu)} \quad .$$

for each $\mu \in \mathbb{E}(\sigma)$. Firstly, we have:

Theorem 2.11: If $\nu \in \widetilde{\mathbb{E}}(\sigma)$ then

$$\mathrm{pls}^*(\mu|\nu) = \overline{p}(\mu|\nu) \tag{2.10}$$

for each $\mu \in \mathbb{E}(\sigma)$.

Proof: We have $K^*(\mu\wedge\nu) = K^*(\mu) \cap K^*(\nu) \subset K^*(\nu)$. So, $\mu\wedge\nu \in \widetilde{\mathbb{E}}(\sigma)$. Taking into account the theorems 2.8., 2.9. and 2.10. we get $\mathrm{pls}^*(\mu\wedge\nu) = \overline{p}(\mu\wedge\nu)$ and $\mathrm{pls}^*(\nu) = \overline{p}(\nu)$. ■

Moreover, we define a Bayespartition as such sequence $\{\nu_n\} \subset \mathbb{E}(\sigma)$ of pairwise F-separated fuzzy subsets that it contains only possible events i.e. $\mathrm{pls}^*(\nu_n) = 0$ for each n and $\sup_n \{\nu_n\}$ is a certain event i.e. $\mathrm{pls}^*(\sup_n \{\nu_n\}) = 1$. Take into account the Bayes method of inference on extended domain $\mathbb{E}(\sigma)$. Each element of Bayes partition $\{\nu_n\}$ will be interpreted as a diagnosis. Furthermore, let $\mu \in \mathbb{E}(\sigma)$ be an imprecise image of symptoms. Since any extension of p on σ to $\widetilde{\mathbb{E}}(\sigma)$ is a fuzzy P-measure, if $\mu \in \widetilde{\mathbb{E}}(\sigma)$ and $\{\nu_n\} \subset \widetilde{\mathbb{E}}(\sigma)$ then the conditional plausibility $\mathrm{pls}^*(\nu_k|\mu)$ is explicitly given by the usual Bayes formula (see [7])

$$\mathrm{pls}^*(\nu_k|\mu) = \overline{p}(\nu_k|\mu) = \frac{\overline{p}(\mu|\nu_k)\cdot\overline{p}(\nu_k)}{\sum_n \overline{p}(\mu|\nu_n)\cdot\overline{p}(\nu_n)} \quad . \tag{2.11}$$

Since fuzzy P-measure p on σ cannot be extended to $\mathbb{E}(\sigma)$ (see [5]) , the Bayes formula (2.11) cannot be generalized to the case $\mu \in \mathbb{E}(\sigma)$ and $\{\nu_n\} \subset \mathbb{E}(\sigma)$. On the other side, from practical point-view, the domain of symptoms should be as possible extensive. The next theorem presents some explicite generalization of the

Bayes formula to the case of a symptom from $\mathbb{E}(\sigma)$.

Theorem 2.12: The identity (2.11) holds for the cases:

- $\mu \in \mathbb{E}(\sigma)$ and $\{\nu_n\} \subset \mathbb{E}^*(\sigma)$;
- $\mu \in \mathbb{E}(\sigma)$ and $\{\nu_n\} \subset \widetilde{\mathbb{E}}(\sigma)$ is finite.

Proof: Note that if $\{\nu_n\} \subset \mathbb{E}^*(\sigma)$ or $\{\nu_n\} \subset \widetilde{\mathbb{E}}(\sigma)$ is finite then $\sup_n \{\nu_n\} \in \widetilde{\mathbb{E}}(\sigma)$. Thus

$$\mathrm{pls}^*(\mu) = \mathrm{pls}^*(\mu \wedge \sup_n \{\nu_n\}) = \overline{p}(\mu \wedge \sup_n \{\nu_n\}) =$$

$$= \sum_n \overline{p}(\mu|\nu_k) \cdot \overline{p}(\nu_n) \ .$$

Therefore, we get

$$\mathrm{pls}^*(\nu_k|\mu) = \frac{\mathrm{pls}^*(\nu_k \wedge \mu)}{\mathrm{pls}^*(\mu)} = \frac{\overline{p}(\nu_k \wedge \mu)}{\sum_n \overline{p}(\mu|\nu_n)\,\overline{p}(\nu_n)} =$$

$$= \frac{\overline{p}(\mu|\nu_k) \cdot \overline{p}(\nu_k)}{\sum_n \overline{p}(\mu|\nu_n) \cdot \overline{p}(\nu_n)} \ . \ \blacksquare$$

Of course, the last theorem can be generalized for the case of such $\{\nu_n\} \subset \widetilde{\mathbb{E}}(\sigma)$ that $\sup_n \{\nu_n\} \subset \widetilde{\mathbb{E}}(\sigma)$. Furthermore, in the general case we have:

Theorem 2.13: If $\mu \in \mathbb{E}(\sigma)$ and $\{\nu_n\} \subset \mathbb{E}(\sigma)$ then

$$\frac{\overline{p}(\mu|\nu_k) \cdot \overline{p}(\nu_k)}{\sum_n \hat{p}(\mu|\nu_n) \cdot \hat{p}(\nu_n)} \leqslant \mathrm{pls}^*(\nu_k|\mu) \leqslant \frac{\hat{p}(\mu|\nu_k) \cdot \hat{p}(\nu_k)}{\sum_n \overline{p}(\mu|\nu_n) \cdot \overline{p}(\nu_n)} \ .$$

Proof: Using the Theorem 2.5 we get

$$\mathrm{pls}^*(\nu_k|\mu) = \frac{\mathrm{pls}^*(\nu_k \wedge \mu)}{\mathrm{pls}^*(\mu)} = \frac{\mathrm{pls}^*(\nu_k \wedge \mu)}{\mathrm{pls}^*(\mu \wedge \sup_n \{\nu_n\})} \ .$$

Thus, by (2.9), we obtain

$$\mathrm{pls}^*(\nu_k|\mu) \leqslant \frac{\hat{p}(\nu_k \wedge \mu)}{\overline{p}(\mu \wedge \sup_n \{\nu_n\})} = \frac{\hat{p}(\mu|\nu_k) \cdot \hat{p}(\nu_k)}{\sum_n \overline{p}(\mu|\nu_n) \cdot \overline{p}(\nu_n)}$$

and

$$\mathrm{pls}^*(\vartheta_k|\mu) \geq \frac{\overline{p}(\vartheta_k \wedge \mu)}{\hat{p}(\mu \wedge \sup_n \{\vartheta_n\})} = \frac{\overline{p}(\mu|\vartheta_k)\cdot\overline{p}(\vartheta_k)}{\hat{p}(\sup_n \{\mu\wedge\vartheta_n\})} \geq$$

$$\geq \frac{\overline{p}(\mu|\vartheta_k)\cdot\overline{p}(\vartheta_k)}{\sum_n \hat{p}(\mu|\vartheta_n)\cdot\hat{p}(\vartheta_n)}$$

because, by the continuity from above of $\hat{p}$ (see [6]) and by the mathematical induction, we get

$$\sum_n \hat{p}(\mu|\vartheta_n)\cdot\hat{p}(\vartheta_n) = \lim_{m\to\infty}\Big(\sum_{n=1}^m \hat{p}(\mu|\vartheta_n)\cdot\hat{p}(\vartheta_n) +$$

$$+ \hat{p}(\sup_{n>m}\{\mu\wedge\vartheta_n\})\Big) \geq \hat{p}(\sup_n \{\mu\wedge\vartheta_n\}) \; . \blacksquare$$

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