SOME FURTHER PROPERTIES ON FUZZY CONNECTEDNESS

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We tie up some loose ends left from our first paper [1] . Three new equivalent conditions for connectedness [1] on completely distributive lattices are given. Some new properties of connectedness are studied. All the elements in the pointless fuzzy unit interval F defined by B. Hutton in [2] must be connected ones. The product theorem of L-fuzzy sets on fuzzy connectedness are also given here.

Let L be a completely distributive lattice. (L,δ) is said to be a completely distributive lattice [1] briefly CDTL if δ is closed under finite suprema and arbitrary infima and $\delta \subseteq L$.

Definition 1. [1] Let (L, δ) be a CDTL. An element C in L is connected in (L, δ) if there are no closed elements $U \neq 0$ and $V \neq 0$ such that $C \leq U \vee V$, $C \wedge U \wedge V = 0$, $C \wedge U \neq 0$ and $C \wedge V \neq 0$. (L, δ) is connected if 1 is connected in (L, δ).

Definition 2. [1] Let (L, δ) be a CDTL, and C an element in L. Two elements D and E form a C-separation in (L, δ) iff $C \wedge D \neq 0$, $C \wedge E \neq 0$ and $C \wedge D^{-} \wedge E = C \wedge D \wedge E^{-} = 0$. D and E form a separation in (L, δ) if they form a 1-separation in (L, δ) and at the same time we say that the D is separatable from the E in (L, δ).

Theorem 1. Let (L, δ) be a CDTL and C an element in L. Then C is connected in (L, δ) if and only if there are two closed elements $U \neq 0$ and $V \neq 0$ in (L, δ) such that

$$C \leq U \vee V,\ C \wedge U \wedge V = 0,\ C \not\leq U \text{ and } C \not\leq V.$$

Theorem 2. If (L, δ) is a CDTL, and C and D and E are three elements in L. Then the following three conditions are equivalent:

(1) D and E form a C-separation in (L, δ).

(2) $C \leq D \vee E$, $C \wedge D^- \wedge E = C \wedge D \wedge E^- = 0$, $C \not\leq D^-$ and $C \not\leq E^-$.

(3) $C \leq D \vee E$, $C \wedge D^- \wedge E = C \wedge D \wedge E^- = 0$, $C \not\leq D$ and $C \not\leq E$.

From the above two theorems, we can prove the following conditions 4 —— 6 . (The others were proved in [1])

Theorem 3. If (L, δ) is a CDTL, and C an element in L. Then the following six conditions are equivalent:

(1) C is connected in (L, δ).

(2) For each two molecules a and b and for every mapping

$$h: M(L) \longrightarrow \{ R(e): e \text{ is in } M(C) \}$$

with $h(e) \in R(e)$ for each e in M(C), there exist finite molecules $e_1 = a, e_2, \ldots, e_{n-1}, e_n = b$ such that

$$C \not\leq h(e_i) \vee h(e_{i+1}) \quad \text{for } i=1,2, \ldots, n-1.$$

(3) There exists no C-separation D and E in (L, δ) such that $C \leq D \vee E$.

(4) There are no closed elements $U \neq 0$ and $V \neq 0$ in (L, δ) such that $C \leq U \vee V$, $C \wedge U \wedge V = 0$, $C \not\leq U$ and $C \not\leq V$.

(5) There are no elements D and E in (L, δ) such that $C \leq D \vee E$, $C \wedge D^- \wedge E = C \wedge D \wedge E^- = 0$, $C \not\leq D$ and $C \not\leq E$.

(6) There are no elements D and E in (L, δ) such that

$C \leq D \vee E$, $C \wedge D^- \wedge E = C \wedge D \wedge E^- = 0$, $C \nleq D^-$ and $C \nleq E^-$.

Corollary 3.1. If C is connected in (L, δ) and D and E form a C-separation in (L, δ), then $C \leq D \vee E$ implies that $C \leq D$ or $C \leq E$.

In [1], we proved the following Lemma 1 about the connectedness of the supremum of a family of connected elements in a CDTL (L, δ). Now, we improve the result as follows.

Lemma 1. Let $\{C_i: i \in I\}$ be a family of connected elements in a CDTL (L, δ) satisfying that $C_j \wedge C_k \neq 0$ for each j and k in I, Then $C = \bigvee\{C_i: i \in I\}$ is also connected.

Theorem 4. Let $\{C_i: i \in I\}$ be a family of connected elements in (L, δ). If $C_j^- \wedge C_k \neq 0$ or $C_j \wedge C_k^- \neq 0$ holds for each j and k in I, then $C = \bigvee\{C_i: i \in I\}$ is connected in (L, δ).

Corollary 4.1. For a family $\{C_i: i \in I\}$ of connected elements in (L, δ), if there is some k in I such that C_k is not separatable from any member in $\{C_i: i \in I\}$, then $\bigvee\{C_i: i \in I\}$ must be connected.

Theorem 5. Suppose U and V are two closed elements in (L, δ). If both $U \vee V$ and $U \wedge V$ are connected in (L, δ), then both U and V are connected.

B. Hutton introduced the pointless fuzzy unit interval F in order to study the compacification theory for fuzzy topology. Here , we shall give the result that all the elements in F are connected under any topology given on F.

Lemma 2. Let U and V are two elements in the pointless unit interval F [2]. Then $U \vee V = 1$ implies that U = 1 or V = 1, in which 1 is the largest element in F.

Theorem 6. Any element in the pointless fuzzy unit interval F must be a connected element in F.

Theorem 7. For arbitrary topology δ on F, any element in F is always a connected element in (F, δ).

Theorem 7. The L-fuzzy product space ($\prod X_i$, L, $\prod \tau_i$) of $\{ (X_i$, L, $\tau_i): i \in I \}$ is connected iff each L-fuzzy topological space (X_i, L, τ_i) is connected.

References

[1] X.D.Zhao, Connectedness on fuzzy topological spaces, FSS, 20 (1986), 223 - 240.

[2] B.Hutton, Compacification theory for fuzzy topological spaces, Fuzzy Math. 1 (1983), 35 - 44.

[3] P.M.Pu and Y.M.Liu, Fuzzy topology II, JMAA 77 (1980), 20 - 31.