ELEMENTS OF FUZZY CONVEXITY IN LINEAR SPACES*

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Abstract In this paper we introduce (T,τ)-fuzzy sets, where τ means "linear" or "affine" or "convex" or "cone" and T is any fixed t-norm , in order to emphasize some common properties of these types of fuzzy sets. Some results analogous to the classical (crisp) case are presented for (T,τ)-fuzzy sets and (T,τ)-hulls of fuzzy sets. Because of using any t-norm instead of "min", our results can be viewed as generalizations of known properties for so-called affine and convex fuzzy sets, fuzzy cones and fuzzy subspaces.

Key words: fuzzy point, t-norm, (T,τ)-fuzzy set, star-shaped set

1. Introduction

In his classical paper [6] , Zadeh developed a theory for convex fuzzy sets. There exist many contributions to this theory, see e.g. [1,2,3] and further references in these papers. Recently , Yu Yandong generalized Zadeh's definition in [5]. He introduced and studied T-convex fuzzy sets, where T is any t-norm and he used fuzzy points (in the sense of [4]) as a tool. In the present paper we study T-linear, T-affine, T-convex fuzzy sets and fuzzy T-cones. After some preliminary concepts and properties, we introduce (T,τ)-fuzzy sets, where τ denotes an

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arbitrary (but fixed) word among "linear", "affine", "cone" and "convex" in order to emphasize their common peculiarities. The main properties of (T,τ)-fuzzy sets are listed in Theorem 3.1. In Section 4 we define the (T,τ)-hull of a fuzzy set and we give some results analogous to the crisp case.
If we define two operations for fuzzy sets by $\mathcal{A} \vee \mathcal{B} = \text{conv}(\mathcal{A} \cup \mathcal{B})$ and $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \cap \mathcal{B}$, we can give some lattice theoretic results in Section 5. (We use the same symbol to denote a fuzzy set and its membership function, e.g. $\mathcal{A}$ or $\mathcal{B}$.)
Finally, we prove that any T-affine fuzzy set is identical to a translate of a fuzzy T-subspace.

2.Preliminaries

Let X and Y be two given vector spaces over the field of real numbers **R** and denote F(X) and F(Y) the families of all fuzzy sets on X and Y, respectively.
$x_\lambda \in F(X)$ is a *fuzzy point* if

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases},$$

where $\lambda \in (0,1]$.
A fuzzy point x_λ is an *element of* the fuzzy set $\mathcal{A}$ if $\mathcal{A}(x) \geqq \lambda$. This will be denoted by $x_\lambda \in \mathcal{A}$. Let $\mathcal{A}$, $\mathcal{B} \in F(X)$. Then $\mathcal{A} \subseteq \mathcal{B}$ if $x_\lambda \in \mathcal{A} \Rightarrow x_\lambda \in \mathcal{B}$.
Let T be any fixed t-norm, i.e., a function $T:[0,1]\times[0,1]\to[0,1]$ such that

(i) $T(0,0) = 0$, $T(a,1) = a$
(ii) $T(a,b) = T(b,a)$
(iii) $T(a,b) \leqq T(c,d)$ when $a \leqq c$, $b \leqq d$
(iv) $T(T(a,b),c) = T(a,T(b,c))$.

T is said to be a *positive t-norm* if $T(a,b) > 0$ when $a,b > 0$.

In [5] Yu Yandong introduced the sum of two fuzzy set and the product of a fuzzy set by a real number in the following way:

$$(\mathcal{A} + \mathcal{B})(x) := \sup_{v\in X} T(\mathcal{A}(v), \mathcal{B}(x-v)) \quad \text{for all} \quad x \in X$$

and

$$(a\cdot\mathcal{A})(x) := \begin{cases} \mathcal{A}(a^{-1}x) & \text{if } a \neq 0 \\ \sup_{v\in X} \mathcal{A}(v) & \text{if } a = 0,\ x = 0 \\ 0 & \text{otherwise} \end{cases}$$

It is easy to prove the following properties (see [5]).

Lemma 2.1. Let $\mathcal{A},\mathcal{B},\mathcal{C},x_\lambda,y_\mu,z_\eta \in F(X)$, $a,b,c \in \mathbf{R}$, $c \neq 0$. Then

(1) $ax_\lambda + by_\mu = (ax + by)_{T(\lambda,\mu)}$

(2) $(x_\lambda + y_\mu) + z_\eta = x_\lambda + (y_\mu + z_\eta)$

(3) $\mathcal{A} + \mathcal{B} = \mathcal{B} + \mathcal{A}$

(4) $(\mathcal{A} + \mathcal{B}) + \mathcal{C} = \mathcal{A} + (\mathcal{B} + \mathcal{C})$ if T is continuous

(5) $c(\mathcal{A} + \mathcal{B}) = c\mathcal{A} + c\mathcal{B}$

(6) $a(b\mathcal{A}) = (ab)\mathcal{A}$

(7) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow a\mathcal{A} \subseteq a\mathcal{B}$

(8) $\mathcal{A} + 0\mathcal{B} \subseteq \mathcal{A}$

(9) $\mathcal{A} \subseteq \mathcal{C}$, $\mathcal{B} \subseteq \mathcal{D} \Rightarrow \mathcal{A} + \mathcal{B} \subseteq \mathcal{C} + \mathcal{D}$.

3.(T,τ)-fuzzy sets

Let $\overset{1}{x}_{\lambda_1}, \dots, \overset{s}{x}_{\lambda_s} \in F(X)$ be fuzzy points. The fuzzy point

$a_1\overset{1}{x}_{\lambda_1} + \dots + a_s\overset{s}{x}_{\lambda_s}$ is called

a) a *T-linear combination* of $\overset{1}{x}_{\lambda_1}, \dots, \overset{s}{x}_{\lambda_s}$ if $a_1,\dots,a_s \in \mathbf{R}$,

b) a *T-affine combination* of $\overset{1}{x}_{\lambda_1}, \dots, \overset{s}{x}_{\lambda_s}$ if $a_1,\dots,a_s \in \mathbf{R}$

and $\sum_{i=1}^{s} a_i = 1$;

c) a *T-cone combination* of $\overset{1}{x}_{\lambda_1}, \dots, \overset{s}{x}_{\lambda_s}$ if $a_1 \geqq 0,\dots,a_s \geqq 0$;

d) a *T-convex combination* of $x^1_{\lambda_1}, \ldots, x^s_{\lambda_s}$ if $a_1 \geqq 0, \ldots, a_s \geqq 0$

and $\sum_{i=1}^{s} a_i = 1$.

We can formulate some common properties of these combinations. For this reason we can speak about a *(T,τ)-combination* , where τ is an arbitrary but fixed word among "linear", "affine", "cone" and "convex".

We say that $\mathcal{A} \in F(X)$ is closed under a (T,τ)-combination if for a finite number of fuzzy points being in $\mathcal{A}$, their (T,τ)-combinations are in $\mathcal{A}$, too.

$\mathcal{A} \in F(X)$ is a *(T,τ)-fuzzy set* if it is closed under (T,τ)-combination.A T-linear fuzzy set is called a *fuzzy T-subspace*, a T-cone fuzzy set is called a *fuzzy T-cone*.

For the sake of simplicity, we write only τ instead of (T,τ) when this is unmistakable.

Remark 3.1. It is obvious that every crisp τ-set can be viewed as a (T,τ)-fuzzy set. Moreover, every (min,τ)-fuzzy set is (T,τ)-fuzzy set for any T t-norm.

Lemma 3.1. $\mathcal{A} \in F(X)$ is a τ-fuzzy set if and only if $\mathcal{A}$ is closed under two-points τ-combinations.

Proof. The necessity is obvious. The sufficiency one can prove by induction.

For $x,y \in X$, let $x\tau y$ be denote an arbitrary τ-combination of x,y.

Lemma 3.2. $\mathcal{A} \in F(X)$ is a τ-fuzzy set if and only if

$$\mathcal{A}(x\tau y) \geqq T(\mathcal{A}(x),\mathcal{A}(y))$$

for every $x,y \in X$.

Proof. It is easy.

Theorem 3.1. (a) Let X, Y be vector spaces over **R** and $f:X \to Y$ be a linear mapping. If $\mathcal{B} \in F(Y)$ is a τ-fuzzy set then $f^{-1}[\mathcal{B}]$ is a τ-fuzzy set on X. If T is continuous and

$\mathcal{A} \in F(X)$ is a τ-fuzzy set then $f[\mathcal{A}]$ is a τ-fuzzy set on Y.

(b) Let $\{\mathcal{A}_\gamma \; ; \; \gamma \in \Gamma\}$ be a family of τ-fuzzy sets on X. Then $\bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma$ is a τ-fuzzy set on X.

(c) If $\{\mathcal{A}_\gamma \; ; \; \gamma \in \Gamma\}$ is a chain of τ-fuzzy sets on X then $\bigcup_{\gamma\in\Gamma} \mathcal{A}_\gamma$ is a τ-fuzzy set on X.

(d) If $\mathcal{A},\mathcal{B} \in F(X)$ are τ-fuzzy sets and T is continuous then $\mathcal{A} + \mathcal{B}$ is a τ-fuzzy set on X.

Proof. One can verify (a),(b) and (d) in the same way as in [5].

(c) Denote $\bigcup_{\gamma\in\Gamma} \mathcal{A}_\gamma$ by $\mathcal{D}$ and let $x,y \in X$. Then, by continuity of T , for every fixed $\epsilon > 0$ there exists $\delta > 0$ such that, for every $\lambda \in (\mathcal{D}(x)-\delta,\mathcal{D}(x)+\delta)\cap[0,1]$ and $\mu \in (\mathcal{D}(y)-\delta,\mathcal{D}(y)+\delta)\cap[0,1]$ we have $T(\lambda,\mu) > T(\mathcal{D}(x),\mathcal{D}(y)) - \epsilon$. By definition of $\mathcal{D}$, there exist $\mathcal{A}_i$ and $\mathcal{A}_j \in F(X)$ such that $\mathcal{A}_i(x) \geqq \mathcal{D}(x)-\delta$ and $\mathcal{A}_j(y) \geqq \mathcal{D}(y)-\delta$. $\{\mathcal{A}_\gamma \; ; \; \gamma \in \Gamma\}$ is a chain, so $\mathcal{A}_i \supseteq \mathcal{A}_j$ or $\mathcal{A}_i \subseteq \mathcal{A}_j$. Assume that $\mathcal{A}_i \supseteq \mathcal{A}_j$.Hence we get the following inequalities:

$\bigcup_{\gamma\in\Gamma} \mathcal{A}_\gamma \; (x\tau y) \geqq \mathcal{A}_i(x\tau y) \geqq T(\mathcal{A}_i(x),\mathcal{A}_i(y)) \geqq T(\mathcal{A}_i(x),\mathcal{A}_j(y)) \geqq$

$\geqq T(\bigcup_{\gamma\in\Gamma} \mathcal{A}_\gamma(x), \bigcup_{\gamma\in\Gamma} \mathcal{A}_\gamma(y)) - \epsilon$. From this we get the assertion.

We introduce the following notations:

$$w_r(\mathcal{A}) = \{ \; x \in X \; ; \; \mathcal{A}(x) \geqq r \; \} \; ,$$

$$\sigma_r(\mathcal{A}) = \{ \; x \in X \; ; \; \mathcal{A}(x) > r \; \} \; ,$$

where $\mathcal{A} \in F(X)$ and $r \in [0,1]$.

A crisp set $Y \subset X$ is said to be *star-shaped* relative to a point $x' \in X$ if for every $y \in Y$ and each $\lambda \in [0,1]$ we have that $\lambda x' + (1-\lambda)y \in Y$.

Theorem 3.3. Suppose that $\mathcal{A} \in F(X)$ is T-convex. If there exists an $x' \in X$ such that $\mathcal{A}(x) = T(\mathcal{A}(x'),\mathcal{A}(x))$ for all $x \in X$ then $\sigma_r(\mathcal{A})$ and $w_r(\mathcal{A})$ are star-shaped relative to x' for every $r \in [0,1]$.

Proof. Assume that $y \in w_r(\mathcal{A})$. The T-convexity of $\mathcal{A}$ implies that $\mathcal{A}(\lambda x' + (1-\lambda)y) \geqq T(\mathcal{A}(x'),\mathcal{A}(y)) = \mathcal{A}(y) \geqq r$, i.e. , $\lambda x' + (1-\lambda)y \in w_r(\mathcal{A})$.

For $\sigma_r(\mathcal{A})$ the proof is similar.

Corollary 3.1. If $\mathcal{A} \in F(\mathbf{R})$ and there exists an $x' \in \mathbf{R}$ such that $\mathcal{A}(x') = 1$, then $\mathcal{A}$ is T-convex if and only if $T = \min$.

Proof. In this case $w_r(\mathcal{A})$ is star-shaped relative to x' for every $r \in [0,1]$.This is true if and only if $w_r(\mathcal{A})$ is convex , i.e., if and only if T = min.

Remark 3.2. Let $h(\mathcal{A}) = \sup_{x \in X} \mathcal{A}(x)$. A simple sufficient condition for $\mathcal{A}$ to be a T-convex fuzzy set is the following:

if $\mathcal{A}(x) \geqq T(h(\mathcal{A}),h(\mathcal{A}))$ for every $x \in X$ then $\mathcal{A}$ is T-convex. Moreover, it is easy to prove that if $\mathcal{A} \in F(X)$ is a τ-fuzzy set then $w_1(\mathcal{A})$ is a (crisp) τ-set and if T is positive then $\sigma_0(\mathcal{A})$ is also a (crisp) τ-set.

4. (T,τ)-hulls of fuzzy sets

If $\mathcal{A} \in F(X)$ then the *(T,τ)-hull of $\mathcal{A}$* is the smallest (T,τ)-fuzzy set on X including $\mathcal{A}$. We denote by $\mathrm{lin}(\mathcal{A})$, $\mathrm{aff}(\mathcal{A})$, $\mathrm{con}(\mathcal{A})$ and $\mathrm{conv}(\mathcal{A})$ the linear, affine, cone and convex hulls of $\mathcal{A}$, respectively.

Theorem 4.1. Assume that T is continuous and $\mathcal{A} \in F(X)$. Then

(a) $$\mathrm{lin}(\mathcal{A}) = \bigcup \left\{ \sum_{i=1}^{s} a_i x^i_{\lambda_i} \; ; \; x^i_{\lambda_i} \in \mathcal{A} \, , \; a_i \in \mathbf{R} \, , \; s \in \mathbf{N} \right\}$$

(b) $$\mathrm{aff}(\mathcal{A}) = \bigcup \left\{ \sum_{i=1}^{s} a_i x^i_{\lambda_i} \; ; \; x^i_{\lambda_i} \in \mathcal{A} \, , \; a_i \in \mathbf{R}, \; \sum_{i=1}^{s} a_i = 1, \; s \in \mathbf{N} \right\}$$

(c) $$\mathrm{con}(\mathcal{A}) = \bigcup \left\{ \sum_{i=1}^{s} a_i x^i_{\lambda_i} \; ; \; x^i_{\lambda_i} \in \mathcal{A} \, , \; a_i \geqq 0 \, , \; s \in \mathbf{N} \right\}$$

(d) $\operatorname{conv}(\mathcal{A}) = \bigcup \left\{ \sum_{i=1}^{s} a_i x^i_{\lambda_i} \; ; \; x^i_{\lambda_i} \in \mathcal{A} \,,\; a_i \geqq 0 \,, \sum_{i=1}^{s} a_i = 1,\; s \in \mathbf{N} \right\}.$

Proof. Let us denote the right-hand side of (a) , (b) , (c) and (d) by $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_3$, $\mathcal{B}_4$, respectively.It is obvious that $\mathcal{A} \subseteq \mathcal{B}_1 \subseteq \operatorname{lin}(\mathcal{A})$, $\mathcal{A} \subseteq \mathcal{B}_2 \subseteq \operatorname{aff}(\mathcal{A})$, $\mathcal{A} \subseteq \mathcal{B}_3 \subseteq \operatorname{con}(\mathcal{A})$ and $\mathcal{A} \subseteq \mathcal{B}_4 \subseteq \operatorname{conv}(\mathcal{A})$. Hence it is sufficient to show that $\mathcal{B}_1$ is linear, $\mathcal{B}_2$ is affine, $\mathcal{B}_3$ is cone and $\mathcal{B}_4$ is convex. We prove the assertion only for $\mathcal{B}_4$. The remainder is analogous.
Let x_λ , $y_\mu \in \mathcal{B}_4$ and $a \in [0,1]$.Suppose that $\lambda \neq 0, \mu \neq 0$.(It is obvious that $ax_\lambda + (1-a)y_\lambda \in \mathcal{B}_4$ when $\lambda = 0$ or $\mu = 0$.) Let $\epsilon > 0$ be fixed. Since T is continuous , there exist λ_0 and μ_0 such that $0 \leqq \lambda_0 < \lambda$ and $0 \leqq \mu_0 < \mu$ and $T(\lambda_0,\mu_0) > T(\lambda,\mu) - \epsilon$. By definition of $\mathcal{B}_4$, there exist $x_\eta, y_\xi \in \mathcal{B}_4$ such that

$$x_\eta = \sum_{i=1}^{m} a_i x^i_{\lambda_i} \quad , \quad y_\xi = \sum_{i=1}^{n} b_i y^i_{\mu_i} \quad \text{and} \quad \lambda_0 \leqq \eta \leqq \lambda \; , \mu_0 \leqq \xi \leqq \mu \; ,$$

where $a_i, b_i \geqq 0$ and $x^i_{\lambda_i}$, $y^j_{\mu_j} \in \mathcal{B}_4$ .It is evident that

$ax_\eta + (1-a)y_\xi \in \mathcal{B}_4$, i.e., $(ax + (1-a)y)_{T(\eta,\xi)} \in \mathcal{B}_4$,whence we get that $\mathcal{B}_4(ax + (1-a)y) \geqq T(\eta,\xi) \geqq T(\lambda_0,\mu_0) > T(\lambda,\mu) - \epsilon$.So (because $\epsilon > 0$ is arbitrary) it follows that $\mathcal{B}_4$ is convex. Thus our theorem is proved.

Theorem 4.2. Let us denote by $\tau(\mathcal{A})$ the τ-hull of $\mathcal{A} \in F(X)$. Then the following properties are hold:

(a) $\tau(\tau(\mathcal{A})) = \tau(\mathcal{A})$

(b) $\mathcal{A} \subseteq \mathcal{B} \Rightarrow \tau(\mathcal{A}) \subseteq \tau(\mathcal{B})$

(c) $\tau(\bigcap_{\gamma\in\Gamma} \mathcal{A}_\gamma) \subseteq \bigcap_{\gamma\in\Gamma} \tau(\mathcal{A}_\gamma)$

(d) $\bigcup_{\gamma\in\Gamma} \tau(\mathcal{A}_\gamma) \subseteq \tau(\bigcup_{\gamma\in\Gamma} \mathcal{A}_\gamma)$

(e) $\bigcup_{\gamma\in\Gamma} \tau(\mathcal{A}_\gamma) = \tau(\bigcup_{\gamma\in\Gamma} \mathcal{A}_\gamma)$ if and only if $\bigcup_{\gamma\in\Gamma} \tau(\mathcal{A}_\gamma)$ is a τ-fuzzy set.

Proof. Parts (a) and (b) are obvious. Parts (c), (d) and (e) follow from (a) and (b).

5. The lattice of T-convex fuzzy sets

Lemma 5.1. If $\mathcal{A}, \mathcal{B} \in F(X)$ are τ-fuzzy sets , then $\operatorname{conv}(\mathcal{A} \cup \mathcal{B}) = \tau(\mathcal{A} \cup \mathcal{B})$, where τ = con or τ = lin now.

Proof. It is obvious that $\operatorname{conv}(\mathcal{A} \cup \mathcal{B}) \subseteq \tau(\mathcal{A} \cup \mathcal{B})$.

If $x_\lambda \in \tau(\mathcal{A} \cup \mathcal{B})$ then $x_\lambda = \sum_{i=1}^{s} b_i x^i_{\lambda_i}$ for some $s \in \mathbf{N}$, $x^i_{\lambda_i} \in \mathcal{A} \cup \mathcal{B}$ $(i=1,\dots,s)$ and $b_1,\dots,b_s$ are real numbers according to the τ-combination.We can make two groups from $x^i_{\lambda_i}$'s: $x^i_{\lambda_i}$ belongs to the first group if $x^i_{\lambda_i} \in \mathcal{A}$ and $x^i_{\lambda_i}$ belongs to the second group if $x^i_{\lambda_i} \in \mathcal{B}$ but $x^i_{\lambda_i} \notin \mathcal{A}$. For the sake of simplicity assume that $x^1_{\lambda_1},\dots,x^k_{\lambda_k}$ form the first group $(k \leqq s)$ and $x^{k+1}_{\lambda_{k+1}},\dots,x^s_{\lambda_s}$ form the second group. Let us define two fuzzy points by the following way:

$$y^1_\mu = \begin{cases} 2\sum_{i=1}^{k} b_i x^i_{\lambda_i} & \text{if } 0< k \leqq s \\ 0_\mu & \text{if } k=0 \end{cases}$$

$$y^2_\eta = \begin{cases} 2 \sum_{i=k+1}^{s} b_i x^i_{\lambda_i} & \text{if } k < s \\ 0_\eta & \text{if } k = s. \end{cases}$$

Then $y^1_\mu \subseteq \mathcal{A}$, $y^2_\eta \subseteq \mathcal{B}$ because $\mathcal{A}$ and $\mathcal{B}$ are τ-fuzzy sets.

We can write that $x_\lambda = \frac{1}{2} y^1_\mu + \frac{1}{2} y^2_\eta$, whence we get that $\tau(\mathcal{A} \cup \mathcal{B}) \subseteq \text{conv}\ (\mathcal{A} \cup \mathcal{B})$.

Denote by $F_{conv}(X)$, $F_{cone}(X)$, $F_{lin}(X)$ the all T-convex, T-cone and T-linear fuzzy subsets of X, respectively.

Theorem 5.1. Let us define two operations on F(X) by $\mathcal{A} \vee \mathcal{B} = \text{conv}\ (\mathcal{A} \cup \mathcal{B})$ and $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \cap \mathcal{B}$. Then $F_{conv}(X)$ is a lattice with null element $\mathbb{O}(x) \equiv 0$ and with unit element $\mathbb{1}(x) \equiv 1$; furthermore, $F_{cone}(X)$ is a sublattice of $F_{conv}(X)$ and $F_{lin}(X)$ is a sublattice of $F_{cone}(X)$.

Proof. One can easy verify that the lattice axioms hold for $F_{conv}(X)$. The second assertion follows from the fact that the above-defined operations assign a T-cone to a T-cone and a T-subspace to a T-subspace. In the case of $\wedge$ this is evident and for $\vee$ it follows from Lemma 5.1.

6. T-affine fuzzy sets and fuzzy T-subspaces

Theorem 6.1. $\mathcal{A} \in F(X)$ is a T-affine fuzzy set if and only if it is a translate of a fuzzy T-subspace.

Proof. (a) Let $\mathcal{L}$ be a fuzzy T-subspace and $x_\lambda \in F(X)$ be a fuzzy point. Then $(\mathcal{L} + x_\lambda)(y) = T(\lambda, \mathcal{L}(y-x))$. We can write the following inequalities:

$$(\mathcal{L} + x_\lambda)(ay + (1-a)z + x) = T(\lambda, \mathcal{L}(a(y-x) + (1-a)(z-x)) \geqq$$
$$\geqq T(\lambda, T(\mathcal{L}(y-x), \mathcal{L}(z-x))) = T(T(\lambda, \mathcal{L}(y-x)), \mathcal{L}(z-x)) \geqq$$
$$\geqq T(T(\lambda, \mathcal{L}(y-x))\ ,\ T(\lambda, \mathcal{L}(z-x))) = T((\mathcal{L} + x_\lambda)(y)\ ,\ (\mathcal{L} + x_\lambda)(z)).$$

(b) Let $\mathcal{H}$ be a T-affine fuzzy set and $x_\lambda \subseteq \mathcal{H}$. Assume now that $a \neq 1$. Then $\mathcal{H}(ay + bz + x) = \mathcal{H}(a(y+x) + (1-a)(x + \frac{b}{1-a} z)) \geqq$

$\geqq T(\mathcal{H}(y+x) , \mathcal{H}(x + \frac{b}{1-a} z)) = T(\mathcal{H}(y+x) , \mathcal{H}(\frac{b}{1-a}(z+x) + (1 - \frac{b}{1-a})x))$

$\geqq T(\mathcal{H}(y+x) , T(\mathcal{H}(z+x),\mathcal{H}(x))) \geqq T(\mathcal{H}(y+x),T(\mathcal{H}(z+x),\lambda))$.

If $a = b = 1$ then $\mathcal{H}(y + z + x) = \mathcal{H}((y+x) + (z+x) - x) \geqq$

$\geqq T(\mathcal{H}(y+x),T(\mathcal{H}(z+x),\mathcal{H}(x)))$.

Because $(\mathcal{H} - x_\lambda)(y) = T(\lambda,\mathcal{H}(y+x))$, we can apply the inequalities above: $(\mathcal{H} - x_\lambda)(ay + bz) \geqq T(\lambda,T(\mathcal{H}(y+x),T(\mathcal{H}(z+x),\mathcal{H}(x))) \geqq$

$\geqq T(T(\lambda,\mathcal{H}(y+x)) , T(\lambda,\mathcal{H}(z+x))) = T((\mathcal{H} - x_\lambda)(y) , (\mathcal{H} - x_\lambda)(z))$.

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