FUZZY BOOLEAN ALGEBRA AND ITS PROPERTIES

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The concepts of fuzzy Boolean algebra, normal fuzzy Boolean algebra, and the base of a finite fuzzy Boolean algebra are introduced in this paper. The conditions are discussed under which a fuzzy set can be formed into a fuzzy Boolean algebra and a normal fuzzy Boolean algebra. And the elementary properties of fuzzy Boolean algebra are explored. The necessary and sufficient condition for the existence of the main base of a finite fuzzy Booean algebra is given.

1. Introduction

Since Zadeh introduced fuzzy sets [1], many fuzzy mathematical systems have been proposed and being researched. However, as the classical truth set for fuzzy sets is the unit interval [0,1](see [1] & [2]), and the complementarity law doesn't hold for this interval under the maxi-min priciple, the work for fuzzy Boolean algebras has made little headway. This paper, making use of the concept of fuzzy point [3], presents the concepts of fuzzy Boolean algebra, normal fuzzy Boolean algebra, and the base of a finite fuzzy Boolean algebra. Then the conditions are discussed under which a fuzzy set can be formed into a fuzzy Boolean algebra. And the elementary properties of fuzzy Boolean algebra are explored. The necessary and sufficient condition for the existence of the main base of a finite fuzzy Boolean algebra is given.

2. Fuzzy Boolean algebra

Definition 1. Let $(L, <_L)$ be a lattice. If to each element λ in L corresponds an element λ' such that

$$1)\ \lambda_1 <_L \lambda_2 \Longleftrightarrow \lambda'_2 <_L \lambda'_1, \qquad 2)\ (\lambda')' = \lambda$$

then the lattice is said to be complemented [4].

Definition 2. Let X be a set, and L be a complemented and bounded lattice, denoted by $(L, \cdot, ', +, 0, 1)$ [4]. Then a fuzzy set A of X is characterized by a function $\mu_A : X \rightarrow L$ from X to L, called the membership function.

Definition 3. The support set of a fuzzy set A is

$$S_A = \{x \mid x \in X,\ \mu_A(x) \neq 0\}.$$

Definition 4. Let A and B be both fuzzy sets of a set X. If $\mu_B(x) \leqslant_L \mu_A(x)$ for all $x \in X$, then B is called a subset of A, denoted $B \subseteq A$ for short.

Definition 5. Let A and B be both fuzzy sets of a set X. Then A is said to be equal to B, written $A = B$, if $\mu_A(x) = \mu_B(x)$ for all $x \in X$.

It is obvious that $A = B$ is equivalent to $A \subseteq B$ and $B \subseteq A$.

Definition 6. A fuzzy point a_λ in X is a fuzzy set with membership function

$$\mu(x) = \begin{cases} \lambda, & \text{for } x = a, \\ 0, & \text{otherwise,} \end{cases}$$

where $a \in X$ and $\lambda \in L$. a_λ is said to have support a and value λ.

Obviously, it is true that $a_0 = b_0$ for all $a, b \in X$, and in case α and β are not both the value 0, $a_\alpha \neq b_\beta$ if and only if $a \neq b$ or $\alpha \neq \beta$.

Definition 7. Let A be a fuzzy set of a set X, and a_λ a fuzzy point in X. We say that a_λ belongs to A, or a_λ is in A, denoted $a_\lambda \in A$, iff $a \in S_A$ and $\lambda <_L \mu_A(a)$, where $<_L$ is the partially ordered relation in the lattice L. Particularly, if the fuzzy point $a_{\mu_A(a)}$ is in A, $a_{\mu_A(a)}$ is referred to as a main fuzzy point of A, written $\dot{a}$.

It is evident that $a_{\mu_A(a)} \in A$ iff $\mu_A(a) \neq 0$, i.e. $a \in S_A$. And it is easy to see that $B \subseteq A$ is equivalent to $x_\lambda \in A$ for all $x_\lambda \in B$. In fact, if $B \subseteq A$, then obviously, $S_B \subseteq S_A$, and for all $x_\lambda \in B$, we have $x \in S_B \subseteq S_A$ and $\lambda <_L \mu_B(x) <_L \mu_A(x)$, so that $x_\lambda \in A$. Conversely, if $x_\lambda \in A$ for all $x_\lambda \in B$, then for all $x \in S_B$ we have $x_{\mu_B(x)} \in A$ so $\mu_B(x) <_L \mu_A(x)$. In other hand, for every $x \in (X - S_B)$, we have $\mu_B(x) = 0 <_L \mu_A(x)$. Hence, we have $\mu_B(x) <_L \mu_A(x)$ for any $x \in X$, that is, $B \subseteq A$.

The above results show that in the view of fuzzy point, a fuzzy set may be viewed as a crisp set. The symbols $\subseteq$ and $=$ may be seen as the ordinary inclusion and equality of sets, respectively. And a fuzzy set A may be viewed as a crisp set composed of all fuzzy points of A.

Definition 8. We define the relation $<$ between two fuzzy points x_α and y_β by:

$x_\alpha < y_\beta$ is an abbreviation for $x = y$ and $\alpha <_L \beta$.

Definition 9. Let A be a fuzzy set. A fuzzy operation on A consists in assoiating with each ordered pair (x_α, y_β) of elements of A a third element z_γ of A, written $z_\gamma = x_\alpha \odot y_\beta$, such that

1) for all $x_\eta, y_\theta \in A$, the support of $x_\eta \odot y_\theta$ is z;
2) for all $u_\alpha, v_\beta \in A$, the value of $u_\alpha \odot v_\beta$ is γ.

Definition 10. Let A be a fuzzy set, and let $\times$ and $\oplus$ be two fuzzy operations on A. If for all $x_\alpha, y_\beta, z_\gamma \in A$, the following hold:

1) $x_\alpha * y_\beta = y_\beta * x_\alpha$, and $x_\alpha \oplus y_\beta = y_\beta \oplus x_\alpha$;
2) $x_\alpha * (y_\beta \oplus z_\gamma) = (x_\alpha * y_\beta) \oplus (x_\alpha * z_\gamma)$, and $x_\alpha \oplus (y_\beta * z_\gamma) = (x_\alpha \oplus y_\beta) * (x_\alpha \oplus z_\gamma)$;
3) there exist two fuzzy points c_ε and e_δ in A such that $x_\alpha \oplus c_\varepsilon = x_\alpha$, and $x_\alpha * e_\delta = x_\alpha$;
4) for each main fuzzy point $\dot{u} \in A$, there exists a fuzzy point $v_\lambda \in A$ such that $\dot{u} * v_\lambda > c_\varepsilon$, and $\dot{u} \oplus v_\lambda < e_\delta$, we often write $\sim \dot{u}$ for v_λ,

then $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ is called a fuzzy Boolean algebra.
In particular, if the condition 4) holds true for equalities, namely, $\dot{u} * v_\lambda = c_\varepsilon$, and $\dot{u} \oplus v_\lambda = e_\delta$, then A is called a normal fuzzy Boolean algebra.

Theorem 1. Let $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ be a fuzzy Boolean algebra, and S_A the support set of A. Then S_A may be formed into a Boolean algebra.

Proof. For all $x, y \in S_A$, there are fuzzy points $x_\alpha, y_\beta \in A$. Let $u_\eta = x_\alpha * y_\beta$, and $v_\theta = x_\alpha \oplus y_\beta$. Then $u_\eta \in A$ and $v_\theta \in A$, so $u \in S_A$ and $v \in S_A$, and u, v are uniquely determined by x and y, in other words, u and v are independent of the values α and β. We define $x \cap y = u$ and $x \cup y = v$. Then $\cap$ and $\cup$ are two algebra operations on S_A.

It is not difficult to verify that

i) the operations $\cap$ and $\cup$ satisfy commutative laws and distributive laws;
ii) the elements c and e belong to S_A, and $x \cup c = x$, $x \cap e = x$ for all $x \in S_A$;
iii) for each element $x \in S_A$, there exists a element $y \in S$ such that $x \cap y = c$, and $x \cup y = e$.

Notice that $\dot{x} \in A$ if $x \in S_A$, so there must exist a fuzzy point $y_\lambda \in A$ such that $\dot{x} * y_\lambda > c_\varepsilon$, and $\dot{x} \oplus y_\lambda < e_\delta$. And by Definition 8, iii) holds true.
The element y is referred to as the complement of x, written $\bar{x}$.
Thus, $(S_A, \cap, \cup, -, c, e)$ is a Boolean algebra by Huntington's Axiomatics [5].

Theorem 2. Let A be a fuzzy set, S_A the support set of A, and $(S_A, \cap, \cup, -, c, e)$ a Boolean algebra. And let the operations $\cdot$ and + of the truth set $(L, \cdot, +, ', 0, 1)$ of A be distributive. If for all $x, y \in S_A$, the following hold:

a) $\mu_A(x) \cdot \mu_A(y) <_L \mu_A(x \cap y)$;
b) $\mu_A(x) + \mu_A(y) <_L \mu_A(x \cup y)$;
c) $(\mu_A(x))' <_L \mu_A(\bar{x})$,

then A may be formed into a fuzzy Boolean algebra.

Proof. Let $x_\alpha * y_\beta = (x \cap y)_{\alpha\beta}$, and $x_\alpha \oplus y_\beta = (x \cup y)_{\alpha+\beta}$. For all $x_\alpha, y_\beta \in A$, we have that $x, y \in S_A$, $\alpha <_L \mu_A(x)$, $\beta <_L \mu_A(y)$, and $\alpha \cdot \beta <_L \mu_A(x) \cdot \mu_A(y) <_L \mu_A(x \cap y)$, thus $(x \cap y)_{\alpha\beta} \in A$. Hence $*$ is a fuzzy operation on A. Similarly, $\oplus$ is also a fuzzy operation on A.

It is easy to check that the operations $*$ and $\oplus$ are commutative and distributive. Let's write δ for $\mu_A(e)$. Then $\alpha <_L \mu_A(x) <_L \mu_A(x) + \mu_A(\bar{x}) <_L \mu_A(x \cup \bar{x}) = \mu_A(e) = \delta$ for all $x_\alpha \in A$, so $x_\alpha * e_\delta = (x \cap e)_{\alpha \cdot \delta} = x_\alpha$, and notice that

$$x_\alpha \oplus c_0 = (x \cup c)_{\alpha + 0} = x_\alpha,$$
$$\dot{x} * \bar{x}_{(\mu_A(x))'} = (x \cap \bar{x})_{\mu_A(x) \cdot (\mu_A(x))'} > c_0, \text{ and}$$
$$\dot{x} \oplus \bar{x}_{(\mu_A(x))'} = (x \cup \bar{x})_{\mu_A(x) + (\mu_A(x))'} < e_{\mu_A(x) + \mu_A(\bar{x})} < e_\delta .$$

Define $\sim \dot{x} = \bar{x}_{(\mu_A(x))'}$. Hence $(A, *, \oplus, \sim, c_0, e_\delta)$ is a fuzzy Boolean algebra.

The fuzzy Boolean algebra $(A, *, \oplus, \sim, c_0, e_\delta)$ given in Theorem 2 is referred to as the fuzzy Boolean algebra induced by the support set S_A.

Theorem 3. Let A be a fuzzy set, S_A the support set of A, and $(S_A, \cap, \cup, -, c, e)$, $(L, \cdot, +, ', 0, 1)$ tow Boolean algebras. If for all $x, y \in S_A$, the following hold:

a) $\mu_A(x) \cdot \mu_A(y) <_L \mu_A(x \cap y)$;

b) $\mu_A(x) + \mu_A(y) <_L \mu_A(x \cup y)$;

c) $(\mu_A(x))' <_L \mu_A(\bar{x})$,

then the fuzzy Boolean algebra induced by S_A is normal.

Proof. Since $1 = \mu_A(x) + (\mu_A(x))' <_L \mu_A(x \cup \bar{x}) = \mu_A(e) <_L 1$, we have $\mu_A(e) = 1$, so the fuzzy Boolean algebra induced by S_A is $(A, *, \oplus, \sim, c_0, e_1)$ by Theorem 2.

We can easily see that $(A, *, \oplus, \sim, c_0, e_1)$ is a normal fuzzy Boolean algebra, only noticing that $\mu_A(x) \cdot (\mu_A(x))' = 0$, and $\mu_A(x) + (\mu_A(x))' = 1$ for all $x \in S_A$.

Theorem 4. Let A be a fuzzy set, S_A the support set of A, and $(S_A, \cap, \cup, -, c, e)$, $(L, \cdot, +, ', 0, 1)$ be two Boolean algebras. If

a) $\mu_A(x \cap y) = \mu_A(x) \cdot \mu_A(y)$;

b) $\mu_A(x \cup y) = \mu_A(x) + \mu_A(y)$;

c) $\mu_A(\bar{x}) = (\mu_A(x))'$,

then the set A^* composed of all main fuzzy points in A may be formed into a Boolean algebra, and A^* is isomorphic to S_A.

Proof. The proof of this theorem is simple and will be omitted here.

3. The elementary properties of fuzzy Boolean algebras

Theorem 5. Let $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ be a fuzzy Boolean algebra, and A^* the set composed of all main fuzzy points in A. Then, $(A^*, *, \oplus, \sim, \dot{c}, \dot{e})$ is a Boolean algebra iff the operations $*$ and $\oplus$ are closed on A^*.

Proof. Sufficiency: Because $*$ and $\oplus$ are closed on A^*, they are both the operations on A^*, and they are both commutative and distributive on A^*.

By Definition 10, we have

i) for all $\dot{x} \in A^*$, $\dot{x} * e_\delta = \dot{x}$, and $\dot{x} \oplus c_\varepsilon = \dot{x}$;

ii) for each $\dot{x} \in A^*$, there is a fuzzy point $y_\lambda \in A$ such that $\dot{x} * y_\lambda > c_\varepsilon$, and $\dot{x} \oplus y_\lambda < e_\delta$.

It is easy to deduce from i) that $\dot{x} * \dot{e} = \dot{x}$ and $\dot{x} \oplus \dot{c} = \dot{x}$ for all $\dot{x} \in A^*$. In fact, the fuzzy points $(\dot{x} * e_\delta)$ and $(\dot{x} * \dot{e})$ are both with the same suppport x (see Definition 9), and $*$ is closed on A^*, so $\dot{x}*\dot{e}=\dot{x}$. In the same way, we have $\dot{x}\oplus\dot{c}=\dot{x}$.

We can similarly deduce from ii) that $\dot{x}*\dot{y}=\dot{c}$ and $\dot{x}\oplus\dot{y}=\dot{e}$, writing $\tilde{\dot{x}}$ for $\dot{y}$. Thus for each main fuzzy point $\dot{x}\in A^*$, there is a main fuzzy point $\tilde{\dot{x}}\in A^*$ such that $\dot{x}*\tilde{\dot{x}}=\dot{c}$ and $\dot{x}\oplus\tilde{\dot{x}}=\dot{e}$.

To sum up, $(A^*, *, \oplus, \sim, \dot{c}, \dot{e})$ is a Boolean algebra.

Necessity: Obvious.

Theorem 6. Let $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ be such a normal fuzzy Boolean algebra that $x_\alpha * y_\beta < x_\eta * y_\theta$, and $x_\alpha \oplus y_\beta < x_\eta \oplus y_\theta$ whenever $x_\alpha < x_\eta$ and $y_\beta < y_\theta$. Then the following hold:

a) $x_\alpha \oplus (x_\alpha * y_\beta) < x_\alpha$;

b) $x_\alpha * (x_\alpha \oplus y_\beta) < x_\alpha$;

c) $x_\alpha \oplus (x_\alpha * \dot{y}) = x_\alpha$;

d) $x_\alpha * (x_\alpha \oplus \dot{y}) = x_\alpha$,

where x_α and y_β are generic fuzzy points in A, and $\dot{y}$ is a generic main fuzzy point in A.

Proof. Obviously, $x_\alpha \oplus (x_\alpha * y_\beta) < x_\alpha \oplus (x_\alpha * \dot{y})$, and $x_\alpha * (x_\alpha \oplus y_\beta) < x_\alpha * (x_\alpha \oplus \dot{y})$. Consequently, we only need to prove that c) and d) hold true.

For any $\dot{y}\in A$, we have

$$\dot{y}\oplus e_\delta = (\dot{y}\oplus e_\delta) * e_\delta = (\dot{y}\oplus e_\delta) * (\dot{y}\oplus \sim\dot{y}) = \dot{y}\oplus(e_\delta * \sim\dot{y}) = \dot{y}\oplus\sim\dot{y} = e_\delta$$

since $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ is a normal fuzzy Boolean algebra. Hence

$$x_\alpha \oplus (x_\alpha * \dot{y}) = (x_\alpha * e_\delta) \oplus (x_\alpha * \dot{y}) = x_\alpha * (e_\delta \oplus \dot{y}) = x_\alpha * (\dot{y}\oplus e_\delta) = x_\alpha * e_\delta = x_\alpha,$$

namely, c) holds true.

Dually, we have $\dot{y} * c_\varepsilon = c_\varepsilon$, and $x_\alpha * (x_\alpha \oplus \dot{y}) = x_\alpha$.

Theorem 7. Let $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ be a fuzzy Boolean algebra, and S_A the support set of A. Assume that the fuzzy operations $*$ and $\oplus$ satisfy $x_\alpha * y_\beta = z_{\alpha\cdot\beta}$, and $x_\alpha \oplus y_\beta = w_{\alpha+\beta}$, where $\cdot$ and $+$ are two operations on the lattice, which is the truth set of A. Then the following hold:

a) $(A, *, \oplus, c_\varepsilon, e_\delta)$ is a distributive lattice with the lower bound c_ε and the upper bound e_δ;

b) $x_\alpha * y_\beta > c_\varepsilon$, and $x_\alpha \oplus y_\beta < e_\delta$ iff $y=\bar{x}$, where $-$ is the complementary operation on the Boolean algebra S_A, which is formed in Theorem 1.

Proof. By Theorem 1, $(S_A, \cap, \cup, -, c, e)$ is a Boolean algebra. Thus the operations $*$ and $\oplus$ satisfy

i) associative laws: $x_\alpha * (y_\beta * z_\gamma) = [x\cap(y\cap z)]_{\alpha\cdot(\beta\cdot\gamma)}$
$= [(x\cap y)\cap z]_{(\alpha\cdot\beta)\cdot\gamma} = (x_\alpha * y_\beta) * z_\gamma$;
$x_\alpha \oplus (y_\beta \oplus z_\gamma) = [x\cup(y\cup z)]_{\alpha+(\beta+\gamma)}$
$= [(x\cup y)\cup z]_{(\alpha+\beta)+\gamma} = (x_\alpha \oplus y_\beta) \oplus z_\gamma$;

ii) absorption laws: $x_\alpha * (x_\alpha \oplus y_\beta) = [x\cap(x\cup y)]_{\alpha\cdot(\alpha+\beta)} = x_\alpha$,
$x_\alpha \oplus (x_\alpha * y_\beta) = [x\cup(x\cap y)]_{\alpha+(\alpha\cdot\beta)} = x_\alpha$;

iii) commutative laws: known (see Definition 10);

iv) distributive laws: known (see Definition 10).

Hence $(A, *, \oplus)$ is a distributive lattice [5].

We define the relation $\circledleq$ between two fuzzy points x_α and y_β in A by: $x_\alpha \circledleq y_\beta \Longleftrightarrow x_\alpha * y_\beta = x_\alpha$. It is easy to verify that $x_\alpha \circledleq y_\beta$ is equivalent to $x_\alpha \oplus y_\beta = y_\beta$, and then $(A, \circledleq)$ is a partially ordered lattice [5].

For all $x_\alpha \in A$, $x_\alpha * e_\delta = x_\alpha$, and $x_\alpha \oplus c_\varepsilon = x_\alpha$, so $c_\varepsilon \circledleq x_\alpha \circledleq e_\delta$, i.e. that $(A, \circledleq)$ is a bounded lattice.

So we arrive at the conclusion a).

Next we verify the conclusion b).

For all $x_\alpha \in A$, $x_\alpha = x_\alpha \oplus c_\varepsilon = (x \cup c)_{\alpha+\varepsilon} = x_{\alpha+\varepsilon}$, and $x_\alpha = x_\alpha * e_\delta = (x \cap e)_{\alpha \cdot \delta} = e_{\alpha \cdot \delta}$. Therefore, $\alpha = \alpha + \varepsilon$, and $\alpha = \alpha \cdot \delta$, that is, $\varepsilon <_L \alpha <_L \delta$.

Since $x \cap y = c$, and $x \cup y = e$ iff $y = \bar{x}$, we have $x_\alpha * y_\beta = (x \cap y)_{\alpha\beta} = c_{\alpha\beta} > c_\varepsilon$, and $x_\alpha \oplus y_\beta = (x \cup y)_{\alpha\beta} = e_{\alpha\beta} < e_\delta$ iff $y = \bar{x}$. Hence the conclusion b) holds true.

Definition 11. Let $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ be a fuzzy Boolean algebra, and B a subset of A. If $(B, *, \oplus, \sim, c_\varepsilon, e_\delta)$ is also a fuzzy Boolean algebra, then B is called a fuzzy Boolean subalgebra of A. And if $(B, *, \oplus, \sim, c_\varepsilon, e_\delta)$ is a normal fuzzy Boolean algebra, then B is called a normal fuzzy Boolean subalgebra of A.

Theorem 8. Let $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ be a fuzzy Boolean algebra, and B a subset of A. Then B is a normal fuzzy Boolean subalgebra of A iff

a) $*$ and $\oplus$ are closed on B;

b) for each main fuzzy point $x_{\mu_\bullet \infty} \in B$, there exists a fuzzy point $y_\lambda \in B$ such that $x_{\mu_\bullet \infty} * y_\lambda = c_\varepsilon$, and $x_{\mu_\bullet \infty} \oplus y_\lambda = e_\delta$.

Proof. Sufficiency: Suppose that $*$ and $\oplus$ are closed on B. Then $*$ and $\oplus$ are commutative and distributive on B, as $B \subseteq A$. And c_ε, e_δ belong to B from b). Thus $(B, *, \oplus, \sim, c_\varepsilon, e_\delta)$ is a normal fuzzy Boolean algebra (see Definition 10). Hence B is a normal fuzzy Boolean subalgebra of A.

Necessity: Straightforward.

4. The base of a finite fuzzy Boolean algebra

Definition 12. Let A be a fuzzy Boolean algebra, and S_A the support set of A. If S_A is a finite set, then A is called a finite fuzzy Boolean algebra.

Definition 13. Let $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ be a finite fuzzy Boolean algebra. If there exist a group of fuzzy points $(f_1)_{\lambda_1}, (f_2)_{\lambda_2}, \ldots, (f_n)_{\lambda_n}$ of A such that each main fuzzy point $\dot{x}$ of A may be uniquely expressed as

$\dot{x} = [\dot{a}_1 * (f_1)_{\lambda_1}] \oplus [\dot{a}_2 * (f_2)_{\lambda_2}] \oplus \ldots \oplus [\dot{a}_n * (f_n)_{\lambda_n}]$, where $a_i = c$ or e, $1 < i < n$, then the group of elements is called a base of A. In particular, if all elements in a base of A are main fuzzy points, then the base is called a main base of A.

Theorem 9. Let $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ be a finite fuzzy Boolean algebra, and S_A the support set of A. If A has a base $(f_1)_{\lambda_1}, (f_2)_{\lambda_2}, \ldots, (f_n)_{\lambda_n}$, then S_A

is such a m-dimension Boolean algebra that $m<n$, and all base elements of S_A are in the set $\{f_1, f_2, \ldots, f_n\}$.

Proof. By Theorem 1, S_A may be formed into a Boolean algebra $(S_A, \cap, \cup, -, c, e)$. Let $s_1, s_2, \ldots, s_m$ be the base of S_A. Then each $\dot{s}_i$, $1<i<m$, may be expressed as

$$\dot{s}_i = [\dot{a}_{i1} * (f_1)_{\lambda_1}] \oplus [\dot{a}_{i2} * (f_2)_{\lambda_2}] \oplus \ldots \oplus [\dot{a}_{in} * (f_n)_{\lambda_n}].$$

Thus

$$s_i = (a_{i1} \cap f_1) \cup (a_{i2} \cap f_2) \cup \ldots \cup (a_{in} \cap f_n), \qquad (1)$$

where $a_{ij} = c$ or e, $1<j<n$. Notice that s_i is a minimal element of S_A, so $s_i \neq c$. Therefore, there is $1<j<n$, such that $a_{ij} = e$, and $f_j \neq c$ in (1), i.e. that s_i may be expressed as

$$s_i = \cdots \cup f_j \cup \cdots .$$

Thereby, $f_j <_{S_A} s_i$, where $<_{S_A}$ is the partially ordered relation in the partially ordered lattice $(S_A, <_{S_A})$ determined by the Boolean algebra S_A [5]. It follows from the properties of minimal elements that $s_i = f_j$. Hence, $s_i \in \{f_1, f_2, \ldots, f_n\}$, $i=1,2,\ldots,m$, and so $m<n$, noticing that $s_1, s_2, \ldots, s_m$ differ from each other.

Theorem 10. Let A be a fuzzy set with two fuzzy operations $*$ and $\oplus$, S_A the support set of A, $L^* = \{\mu_A(x) \mid x \in S_A\}$, and A^* the set composed of all main fuzzy points of A. If $(A^*, *, \oplus, \sim, \dot{c}, \dot{e})$ is a Boolean algebra, then both S_A and L^* may be formed into Boolean algebra, and S_A is isomorphic to A^*. In particular, if S_A is finite, then the dimension of A^* is equal to that of S_A, and not less than that of L^*.

Proof. For all $\alpha, \beta \in L^*$, there are main fuzzy points $\dot{x}$ and $\dot{y}$ in A^* such that $\mu_A(x) = \alpha$ and $\mu_A(y) = \beta$, i.e. that $x_\alpha = \dot{x}$, and $y_\beta = \dot{y}$. Let $u_\eta = x_\alpha * y_\beta$, and $v_\theta = x_\alpha \oplus y_\beta$. Because $(A^*, *, \oplus, \sim, \dot{c}, \dot{e})$ is a Boolean algebra, $*$ and $\oplus$ are closed on A^*, so that u_η and v_θ are in A^*. Thereby η and θ belong to L^*. We define the operations $\alpha \wedge \beta$, and $\alpha \vee \beta$ on L^* to be $\alpha \wedge \beta = \eta$, and $\alpha \vee \beta = \theta$, respectively. As $*$ and $\oplus$ are two fuzzy operations on A, it follows from Definition 9 that the valuse η and θ are uniquely determined by α and β. Hence $\wedge$ and $\vee$ are two algebra operations on L^*.

For each $\dot{x} \in A^*$, there is a $\dot{y}$ in A^* such that $\dot{x} * \dot{y} = \dot{c}$, and $\dot{x} \oplus \dot{y} = \dot{e}$, since $(A^*, *, \oplus, \sim, \dot{c}, \dot{e})$ is a Boolean algebra. Let $\neg\mu_A(x) = \mu_A(y)$, $\gamma = \mu_A(c)$, and $\delta = \mu_A(e)$. Then it is easy to check that $(L^*, \wedge, \vee, \neg, \gamma, \delta)$ is a Boolean algebra.

We can similarly form S_A into a Boolean algebra $(S_A, \cap, \cup, -, c, e)$.

We define a one-to-one correspondence of A^* into S_A: $\dot{x} \to x$. Then we obtain at once that S_A is isomorphic to A^*. Hence the dimension of A^* is equal to that of S_A when S_A is finite.

In the same way used in Theorem 9, we can prove that the dimension of A^* is not less than that of L^* when S_A is finite.

Theorem 11. Let $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ be a finite fuzzy Boolean algebra, and S_A

the support set of A. If the fuzzy operations $*$ and $\oplus$ satisfy that $x_\alpha * y_\beta = z_{\alpha\cdot\beta}$, and $x_\alpha \oplus y_\beta = w_{\alpha+\beta}$ for all $x_\alpha, y_\beta \in A$, then A has a main base iff $*$ and $\oplus$ are closed on A^*, where A^* is the set composed of all main fuzzy points in A.

Proof. Necessity: Let $\dot{f}_1, \dot{f}_2, \ldots, \dot{f}_n$ be a main base of A. By Theorem 9, $(S_A, \cap, \cup, -, c, e)$ is a m-dimension Boolean algebra, $m \leq n$, and all base elements of S_A are in the set $\{f_1, f_2, \ldots, f_n\}$. Without loss of generality, we may assume that $f_1, f_2, \ldots, f_m$ is the base of S_A. We take four steps to prove the necessity as following.

First, we prove that for all $x \in S_A$, $\mu_A(c) \leq_L \mu_A(x) \leq_L \mu_A(e)$.

Since $*$ and $\oplus$ are closed on A, we have $\dot{x} \oplus \dot{c} = (x \cup c)_{\mu_A(x)+\mu_A(c)} = x_{\mu_A(x)+\mu_A(c)} \in A$, and $\dot{x} \oplus \dot{e} = (x \cup e)_{\mu_A(x)+\mu_A(e)} = e_{\mu_A(x)+\mu_A(e)} \in A$. Thus $\mu_A(x) + \mu_A(c) \leq_L \mu_A(x)$, and $\mu_A(x) + \mu_A(e) \leq_L \mu_A(e)$. But $\mu_A(x) \leq_L \mu_A(x) + \mu_A(c)$, and $\mu_A(e) \leq_L \mu_A(e) + \mu_A(x)$, as the truth set L of A is a lattice. Hence $\mu_A(x) = \mu_A(x) + \mu_A(c)$, and $\mu_A(e) = \mu_A(e) + \mu_A(x)$, i.e. $\mu_A(c) \leq_L \mu_A(x) \leq_L \mu_A(e)$.

Therefor

$$\dot{x} * \dot{c} = (x \cap c)_{\mu_A(x)\cdot\mu_A(c)} = c_{\mu_A(c)} = \dot{c}; \tag{2}$$

$$\dot{x} \oplus \dot{c} = (x \cup c)_{\mu_A(x) + \mu_A(c)} = x_{\mu_A(x)} = \dot{x}; \tag{3}$$

$$\dot{x} * \dot{e} = (x \cap e)_{\mu_A(x)\cdot\mu_A(e)} = x_{\mu_A(x)} = \dot{x}; \tag{4}$$

$$\dot{x} \oplus \dot{e} = (x \cup e)_{\mu_A(x)+\mu_A(e)} = e_{\mu_A(e)} = \dot{e}. \tag{5}$$

Secondly, we prove m=n. Suppose that m<n. We can obtain from (2)—(5) that

$$\dot{f}_n = (\dot{c} * \dot{f}_1) \oplus (\dot{c} * \dot{f}_2) \oplus \ldots \oplus (\dot{c} * \dot{f}_{n-1}) \oplus (\dot{e} * \dot{f}_n). \tag{6}$$

Obviously, $f_n \neq c$, otherwise $\dot{f}_n$ can be expressed as

$$\dot{f}_n = (\dot{c} * \dot{f}_1) \oplus (\dot{c} * \dot{f}_2) \oplus \ldots (\dot{c} * \dot{f}_{n-1}) \oplus (\dot{c} * \dot{f}_n),$$

it is different from (6), and so this is contradictory to Definition 13.

In other hand, f_n can be expressed as

$$f_n = (a_1 \cap f_1) \cup (a_2 \cap f_2) \cup \ldots \cup (a_m \cap f_m), \quad a_i = c \text{ or } e, \ 1 \leq i \leq m \tag{7}$$

since $f_1, f_2, \ldots, f_m$ is the base of S_A. Let

$$(f_n)_\lambda = (\dot{a}_1 * \dot{f}_1) \oplus (\dot{a}_2 * \dot{f}_2) \oplus \ldots \oplus (\dot{a}_m * \dot{f}_m).$$

Then $\lambda \leq_L \mu_A(f_n)$, as $*$ and $\oplus$ are closed on A. Thus

$$\dot{f}_n = (f_n \cup f_n)_{\lambda+\mu_A(f_n)} = (f_n)_\lambda \oplus \dot{f}_n$$
$$= (\dot{a}_1 * \dot{f}_1) \oplus \ldots \oplus (\dot{a}_m * \dot{f}_m) \oplus (\dot{c} * \dot{f}_{m+1}) \oplus \ldots \oplus (\dot{c} * \dot{f}_{n-1}) \oplus (\dot{e} * \dot{f}_n). \tag{8}$$

Notice that $f_n \neq c$, hence there exists j, $1 \leq j \leq m$, such that $a_j = e$ in (7). Thus (8) differ from (6). This is contradictory to Definition 13. Hence m=n.

For this reason, A may only have one main base, noticing that the Boolean algebra S_A uniquely has one base.

Thirdly, we prove that

$$(\dot{a}_1 * \dot{f}_1) \oplus (\dot{a}_2 * \dot{f}_2) \oplus \ldots \oplus (\dot{a}_n * \dot{f}_n) \in A^* \tag{9}$$

where $a_i = c$ or e, $1 \leq i \leq n$.

Let $x_\alpha = (\dot{a}_1 * \dot{f}_1) \oplus (\dot{a}_2 * \dot{f}_2) \oplus \ldots \oplus (\dot{a}_n * \dot{f}_n)$. Thus

$$x = (a_1 \cap f_1) \cup (a_2 \cap f_2) \cup \ldots \cup (a_n \cap f_n) \tag{10}$$

From Definition 13, $\dot{x}$ can be expressed as

$$\dot{x} = (\dot{b}_1 * \dot{f}_1) \oplus (\dot{b}_2 * \dot{f}_2) \oplus \ldots \oplus (\dot{b}_n * \dot{f}_n).$$

Thus

$$x = (b_1 \cap f_1) \cup (b_2 \cap f_2) \cup \ldots \cup (b_n \cap f_n) \tag{11}$$

Because x is uniquely expressed by the base of S_A, we have $a_i = b_i$, $1 \leq i \leq n$, in (10) and (11). Hence $x_\alpha = \dot{x} \in A^*$.

Finally we prove that $*$ and $\oplus$ are closed on A^* , i.e. that $\dot{x}*\dot{y} \in A^*$, and $\dot{x}\oplus\dot{y} \in A^*$ for all $\dot{x},\dot{y} \in A^*$.

In fact, for all $\dot{x},\dot{y} \in A^*$, $\dot{x}$ and $\dot{y}$ can be respectively expressed as

$$\dot{x}=(\dot{a}_1 * \dot{f}_1)\oplus(\dot{a}_2 * \dot{f}_2)\oplus\ldots\oplus(\dot{a}_n * \dot{f}_n), \text{ and}$$

$$\dot{y}=(\dot{b}_1 * \dot{f}_1)\oplus(\dot{b}_2 * \dot{f}_2)\oplus\ldots\oplus(\dot{b}_n * \dot{f}_n),\ a_i, b_i \in \{c, e\},\ 1<i<n.$$

Thus $\dot{x}\oplus\dot{y}=[(\dot{a}_1 \oplus \dot{b}_1)*\dot{f}_1]\oplus[(\dot{a}_2 \oplus \dot{b}_2)*\dot{f}_2]\oplus\ldots\oplus[(\dot{a}_n \oplus \dot{b}_n)*\dot{f}_n]$. From (3), (5) and (9), we can easily deduce that $\dot{x}\oplus\dot{y} \in A^*$.

From the properties of minimal elements of a Boolean algebra, we have $f_i \cap f_j \neq c$, iff i=j. Thus $(\dot{b}_i * \dot{f}_i)*\dot{x}=(\dot{b}_i * \dot{a}_i * \dot{f}_i)$, $1<i<n$. Hence

$$\dot{x}*\dot{y}=(\dot{a}_1 * \dot{b}_1 * \dot{f}_1)\oplus(\dot{a}_2 * \dot{b}_2 * \dot{f}_2)\oplus\ldots\oplus(\dot{a}_n * \dot{b}_n * \dot{f}_n)$$

From (2), (4) and (9), we have $\dot{x}*\dot{y} \in A^*$.

So the proof of necessity of this theorem is completed.

Sufficiency: Since $*$ and $\oplus$ are closed on A^* , $(A^*, *, \oplus, \sim, \dot{c}, \dot{e})$ is a Boolean algebra by Theorem 5. Thus A^* has a base, denoted $\dot{f}_1, \dot{f}_2, \ldots, \dot{f}_n$. Obviously, it is a main base of A as well.

From the proof of Theorem 11, we can see that under the conditions of this theorem, the main base of $(A, *, \oplus, \sim, c_\varepsilon, e_\delta)$ is the base of $(A^*, *, \oplus, \sim, \dot{c}, \dot{e})$, vice versa.

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