THE MINIMUN ROW (COLUMN) SPACE OF FUZZY MATRIX

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ABSTRACT

In this paper the definitions of a full row(column) and the minimun row(column) space of a fuzzy matrix are put forward, And we studied some relations between these concepts and ranks of a fuzzy matrix.

Keywords: The fuzzy matrix of a full row(column). The minimum row(column) space of a fuzzy matrix.

1 FUNDAMENTAL CONCEPTS

Let fuzzy matrices $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$ and $k\in [0,1]$. The sum of the two fuzzy matrices, the scalar product of a number and a fuzzy matrix, and the relation "$\leq$" of two fuzzy matrices are defined respectively as follows:

$$A + B = (a_{ij} + b_{ij})_{m\times n} = (\max\{a_{ij},b_{ij}\})_{m\times n} ;$$

$$kA = (ka_{ij})_{m\times n} = (\min\{k,a_{ij}\})_{m\times n} ;$$

$$A\leq B \text{ iff } \forall i,j,\ a_{ij}\leq b_{ij} .$$

Under the addition and scalar product the set of all n-ary fuzzy row(column) vectors forms a fuzzy semilinear space, denoted by $V_n(V^n)$. Let $\{X_1, \ldots, X_t\} \subseteq V_n(V^n)$. The set W of all linear combination of $X_1, \ldots, X_t$ is a subspace of $V_n(V^n)$, denoted by $W = \langle X_1, \ldots, X_t \rangle$, and W is called a generating subspace by $X_1, \ldots, X_t$. And the cardinal number of W is said to be k, writed by Dim (W) = k.

2 THE FUZZY MATRIX OF A FULL ROW(COLUMN)

Definition 2.1 Let A is a fuzzy $m \times n$ matrix . A is called a full row(column) rank matrix if $\rho_r(A) = m$ ($\rho_c(A) = n$).

In the paper [3] is given the concept of fuzzy relational non-derministic equation.

Definition 2.2 Let $A = (a_{ij})_{m \times n}$ is a non-zero fuzzy matrix . A equation in the form of

$$A_{m \times n} = Y_{m \times t} X_{t \times n} \tag{2.1}$$

is called a fuzzy relational non-deterministic equation of the fuzzy matrix A or non-deterministic equation of A,(where A is known, Y and X are unknown). t in (2.1) is called an index. Y and X such that (2.1) holds are called the solution matrices of the non-deterministic equation of A while index is t.

In the paper [2] Wang Hongxu proved that:

Theorem 2.1 For a $m \times n$ fuzzy matrix A, $\rho_s(A) = s$ if and only if the non-deterministic equation of A has not a solution while index $1 \leqslant t \leqslant s-1$, but it has a solution while index $t = s$.

Proposition 2.1 For a fuzzy matrix A of order $n \times n$, if

$$A = B_{n \times s} C_{s \times n} ,$$

then $\rho_s(A) \leq s$.

Theorem 2.2 Let A is a $m \times n$ fuzzy matrix .

(i) If every column vector of A is a linear combination of m-ary fuzzy column vectors $D_1, \ldots, D_k$, then $\rho_s(A) \leq k$.

(ii) If every row vector of A is a linear combination of n-ary fuzzy row vectors $C_1, \ldots, C_k$, then $\rho_s(A) \leq k$.

Proof Since every column vectoe of A is a linear combination of m-ary fuzzy column vectors $D_1, \ldots, D_k$, there exist

$$F_{k \times n} = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \cdots & \cdots & \cdots \\ f_{k1} & \cdots & f_{kn} \end{pmatrix}$$

such that $A = (D_1 \ \ldots \ D_k)_{m \times k} F_{k \times n}$.

Then $(D_1 \ \ldots \ D_k)_{m \times k}$ and $F_{k \times n}$ are solution matrices of the non-deterministic equation of A for $t = k$.

Therefore $\rho_s(A) \leq k$.

Analogously may prove (ii).

Theorem 2.3 Let $\rho_s(A) = s$ and

$$A = B_{m \times s} C_{s \times n} . \tag{2.2}$$

And let

$$B_{m \times s} = (B_1 \ \ldots . \ B_s). \tag{2.3}$$

Then

(i) $B_1, \ldots, B_s$ are linear independent .

(ii) B is a full column rank.

(iii) every column vector of A is a linear combination of $B_1, \ldots, B_s$.

(iv) $C(A) \subseteq \langle B_1, \ldots, B_s \rangle$.

(v) $C_1, \ldots, C_s$ are linear independent.

(vi) C is a full row rank.

(vii) every row vector of A is a linear combination of $C_1, \ldots, C_s$.

(viii) $R(A) \subseteq \langle C_1, \ldots, C_s \rangle$.

Proof. (i) we suppose that $B_1, \ldots, B_s$ are linear dependent , Wthout loss of generality, we assume that B_s is a linear combination of $B_1, \ldots, B_{s-1}$:

$$B_s = \begin{pmatrix} b_{1s} \\ \vdots \\ b_{ms} \end{pmatrix} = (B_1 \ \ldots \ B_{s-1}) \begin{pmatrix} k_1 \\ \vdots \\ k_{s-1} \end{pmatrix} .$$

Then

$$(B_1 \ldots B_{s-1} B_s) = (B_1 \ldots B_{s-1}) \begin{pmatrix} 1 & 0 & \ldots & 0 & k_1 \\ 0 & 1 & \ldots & 0 & k_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & k_{s-1} \end{pmatrix}_{(s-1)\times s}$$

Thus
$$A = (B_1 \ \ldots \ B_{s-1} \ B_s)_{m \times s} C_{s \times n}$$
$$= (B_1 \ \ldots \ B_{s-1})_{m \times (s-1)} (K_{(s-1) \times s} C_{s \times n})$$

where

$$K_{(s-1)\times s} = \begin{pmatrix} 1 & 0 & \ldots & 0 & k_1 \\ 0 & 1 & \ldots & 0 & k_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & k_{s-1} \end{pmatrix}_{(s-1)\times s} .$$

May see that $\rho_s(A) \leqslant s-1$, which is in contradiction with the hypothesis that $\rho_s(A) = s$.

Therefore $B_1, \ldots, B_s$ are linear independent.

(ii) B is a full column rank by (i).

(iii) Every column vector of A is linear combination of $B_1, \ldots, B_s$ by (2.2).

(iv) $C(A) \subseteq \langle B_1, \ldots, B_s \rangle$ by (iii).

Similarly may prove (v), (vi), (vii) and (viii).

Theorem 2.4 Let as A as (2.2), (2,3), and (2.4), then

(i) $\mathrm{Dim} \langle B_1, \ldots, B_s \rangle = s$.

(ii) $\mathrm{Dim} \langle C_1, \ldots, C_s \rangle = s$.

3 THE MINIMUM ROW (COLUMN) SPACE OF A FUZZY MATRIX

Let $W_1, W_2 \subseteq V_n(V^n)$, we know that $W_1 \subseteq W_2 \nRightarrow \mathrm{Dim}(W_1) \leq \mathrm{Dim}(W_2)$ in fuzzy mathematics.

Example 3.1 Let

$$A = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 0.7 \\ 0.7 & 0.6 \end{bmatrix}.$$

The paper [2] finds $\mathrm{Dim}\, R(A) = \rho_r(A) = 3$, but $\rho_s(A) = 2$.

Now we give that :

Definition 3.1 For fuzzy matrix A, a subspace with contains R(A) and cardinal number is minimum is called minimum row space of A, denoted by R min (A). Similarly may define C min (A).

By the definition 3.1 we have that:

Proposition 3.1 $R(A) \subseteq R \min (A)$.

Theorem 3.1 Let as A as (2.2), (2.3), and (2.4). Then

(i) $C \min (A) = \langle B_1, \ldots, B_s \rangle$.

(ii) $R \min (A) = \langle C_1, \ldots, C_s \rangle$.

Proof. By theorem 2.3 we have that $C(A) \subseteq \langle B_1, \ldots, B_s \rangle$, Thus $\langle B_1, \ldots, B_s \rangle$ is a subspace contained C(A).

By the theorem 2.4 we have that

$$\mathrm{Dim} \langle B_1, \ldots, B_s \rangle = s \leq \mathrm{Dim}\, C(A).$$

So that $\quad \mathrm{Dim}\, C \min (A) \leq s$.

We say that Dim C min $(A) \geq s$. otherwise, let

$$\text{Dim C min } (A) = k < s,$$

there exist m-ary column vectors $D_1, \ldots, D_k$ such that $\langle D_1, \ldots, D_k \rangle \supseteq C(A)$. Then every column vector of A is a linear combination of $D_1, \ldots, D_k$. By the theorem 2.2 we know that $\rho_s(A) \leq k < s$, which is in contradiction with the hypothesis that $\rho_s(A) = s$. So that

$$\text{Dim C min } (A) \geq s$$

Therefore Dim C min $(A) = s$.

Theorem3.2 If $m \times n$ fuzzy matrix A there exist a rank , then $\rho(A) = \rho_s(A)$.

Proof. Let $\rho(A) = k$, then $\rho_c(A) = k$, there exist m-ary column vectors $D_1, \ldots, D_k$ such that $\langle D_1, \ldots, D_k \rangle = C(A)$. May suppose that

$$A = (D_1, \ldots, D_k)_{m \times k} M_{k \times n} .$$

Then by proposition 2.1

$$\rho_s(A) \leq k = \rho_c(A).$$

Since $\rho_s(A) = s$, let as A as (2.2), (2.3), and(2.4). Then

$$C \min (A) = \langle B_1, \ldots, B_s \rangle .$$

By the definition 3.1, Dim $\langle B_1, \ldots, B_s \rangle \geq$ DimC(A), i.e.

$$\rho_s(A) = s \geq \rho_s(A).$$

Therefore $\rho_s(A) = \rho_c(A)$, i.e. $\rho_s(A) = \rho(A)$.

In [7] Cao Zhiqiang proved that

Theorem 3.3 For any $n \times n$ matrix A, A is a ultimately periodic, i.e. there exist natural number $p \leq q$, such that

$$A^{q+1} = A^p \qquad (3.1)$$

We write $$M = A^p + A^{p+1} + \dots + A^q \tag{3.2}$$

Theorem 3.4 Let as A as (3.1) and as M as (3.2) . Then

(i) $\rho_s(M) \leq \rho_s(A^p)$;

(ii) $\rho_s(M) \leq \rho_s(A)$;

(iii) $\rho_s(M) \leq \rho_s(A^k)$, $(1 \leq k \leq p)$

(iv) $\rho_s(M) \leq \min \{ \rho_s(A), \rho_s(A^2), \dots, \rho_s(A^p) \}$.

Proof (i) let $\rho_s(A^p) = s$, $A^p = X_{n\times s} Y_{s\times n}$, then

$$\begin{aligned} M &= A^p + A^{p+1} + \dots + A^q \\ &= A^p(I_n + A + \dots + A^{q-p}) \\ &= X_{n\times s} (Y_{s\times n}(I_n + A + \dots + A^{q-p})) \end{aligned}$$

thus $\rho_s(M) \leq s = \rho_s(A^p)$.

Similarly may prove (ii), (iii) and (iv).

Theorem 3.5 Let as A as (3.1) and as M as (3.2). Then

(i) $R(M) \subseteq R(A^k)$, $k = 1, \dots, p$

(ii) $R(M) \subseteq R \min (A^k)$, $k=1, \dots, p$

(iii) $C(M) \subseteq C(A^K)$, $K=1, \dots, p$

(iv) $C(M) \subseteq C \min (A^k)$, $k=1, \dots, p$

(v) $R(A^k) \subseteq R(A^{k+1})$, $k=1,2,\dots$

(vi) $R(A^k) \subseteq R \min (A^{k+1})$, $k=1,2,\dots$

(vii) $R(A^k) \subseteq R(A^{k+h})$, $k,h=1,2,\dots$

(viii) $R(A^k) \subseteq R \min (A^{k+h})$, $k,h=1,2,\dots$

(ix) $C(A^k) \subseteq C(A^{k+1})$, $k=1,2,\dots$

(x) $C(A^k) \subseteq C \min (A^{k+1})$, $k=1,2,\dots$

(xi) $C(A^k) \subseteq C(A^{k+h})$, $h,k=1,2,\dots$

(xii) $C(A^k) \subseteq C \min (A^{k+h})$, $k,h=1,2,\dots$

Proof. (i) let a basic of $R(A^k)$ is $A_1^*, \dots, A_r^*$, then there exist $X_{n\ r}$ such that

$$A^k = X_{n\times r}\begin{pmatrix} A_1^* \\ \vdots \\ A_r^* \end{pmatrix}$$

thus $M = A^p + \dots + A^q = (A^{p-k} + \dots + A^{q-k})\, A^k$

$$= ((A^{p-k} + \dots + A^{q-k})\, X_{n\times r})\begin{pmatrix} A_1^* \\ \vdots \\ A_r^* \end{pmatrix}$$

That is $R(M) \subseteq \langle A_1^*, \dots, A_r^* \rangle = R(A^k)$.

(ii) Since $R(A^k) \subseteq R\min(A^k)$, by (i) we give $R(M) \subseteq R\min(A^k)$.

Similarly may prove other results.

Theorem 3.6 Let as A as (3.1) and as M as (3.2). Then

(i) $\operatorname{Dim} R\min(M) \le \operatorname{Dim} R\min(A^k)$, $k=1,\dots,p$

(ii) $\operatorname{Dim} C\min(M) \le \operatorname{Dim} C\min(A^k)$, $k=1,\dots,p$

(iii) $\operatorname{Dim} R\min(A^k) \le \operatorname{Dim} R\min(A^{K+1})$, $k=1,2,\dots$

(iv) $\operatorname{Dim} R\min(A^k) \le \operatorname{Dim} R\min(A^{k+h})$, $k,h=1,2,\dots$

Theorem 3.7 Let as A as (3.1) and as M as (3.2). And let

$$N = I_n + A + \dots + A^{q-p}.$$

Then (i) $A^p N = M = N A^p$ (3.3)

(ii) $\rho_s(M) \le \rho_s(N)$.

Proof. (i) Obviously.

(ii) By (3.3) $A^p N = M$, but by Theorem 3.6 have

$$\rho_s(A^p N I_n) \le \rho_s(N),$$

That is $\rho_s(A^p N) \le \rho_s(N)$ or $\rho_s(M) \le \rho_s(N)$.

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