HX GROUP

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With the devolopment of modern mathematics, set value mapping (from a set into its power set) is playing an increasingly important role so that the upgrade of all kinds of the structures, studying their power structure, such as ordered, topological, measurable structure, etc, have been considered by more and more people. Naturally it is also rather interesting how to consider the upgrade of algebraic structure. First it is the easiest thing to consider the upgrade of group. In this paper, we study the problem and give some basic and interesting results.

1. THE DEFINITION AND EXAMPLES OF HX GROUP

We always assume that G is a group in the paper.

DEFINITION 1.1. In $2^G-\{\emptyset\}$ we define a algebraic operation:

$$AB=\{ab \mid a\in A,\ b\in B\} \tag{1.1}$$

An nonempty set $\mathcal{G}\subset 2^G-\{\emptyset\}$ is called a HX group on G, if $\mathcal{G}$ is a group with respect to the operation (1.1), which its unit element is denoted by E. Especially, a HX group $\mathcal{G}$ is called a regular HX group, if $e\in E$, which e is just the unit element of G.

It is easy to know that E is a subsemigroup of G from $E^2=E$.

A quotient group of G must be a HX group on G and its unit element be a normal subgroup of G.

THEOREM 1.1. If $\mathcal{G}$ is a HX group on G, then

1) $(\forall A\in\mathcal{G})(|A|=|B|)$;

2) $(\forall A,B\in\mathcal{G})(A\cap B\neq\emptyset\Longrightarrow |A\cap B|=|E|)$

Proof. 1) In one respect we have

$AE=A \Longrightarrow (\forall\ a\in A)(aE\subset AE=A) \Longrightarrow |E|=|aE|\leqslant|A|$;

In the other respect we have

$A^{-1}A=E \Longrightarrow (\forall\ b\in A^{-1})(bA\subset A^{-1}A=E) \Longrightarrow |A|=|bA|\leqslant|E|$.

2) First $|A\cap B|\leqslant|A|=|E|$;

Second, $c\in A\cap B \Longrightarrow cE\subset A\cap B \Longrightarrow |E|=|cE|\leqslant|A\cap B|$. Q.E.D.

Since E is a subgroup we have

Problem 1: Can we form a HX group $\mathcal{G}$ on G by using a subsemigroup E of G and E is just the unit element in $\mathcal{G}$?

Let E be a subsemigroup of G. For any $a,b\in G$, if there exists $h\in E$, such that $a=bh$, then a and b are called right semi-modular congruence, denoted by

$$a\doteq b \quad (\text{right semod}E) \qquad (1.2)$$

Clearly $\doteq$ is a transitive relation. Moreover $\doteq$ is reflexive iff $e\in E$.

For any $a\in G$, write

$$aE=\{b\in G \mid b\doteq a\ (\text{right semod}E)\} \qquad (1.3)$$

It is easy to see

$$aE=\{ah \mid h\in E\} \qquad (1.4)$$

We call aE a left quasi-coset of E, and also have the concept of right quasi-coset, of couse, the concept of quasi-coset.

Remark: That $a\in aE$ is not true. For example, let G be the additive group of real numbers and take $E=(0,+\infty)$. Then E is a semigroup of G, but $a\notin a+E=(a,+\infty)$ for any $a\in G$. If we add a condition, the have the fact:

$e\in E \Longrightarrow (\forall\ a\in G)(a\in aE)$; conversely, $(\exists a\in G)(a\in aE) \Longrightarrow e\in E$.

THEOREM 1.2. Let H be a subgroup of G and E be a subset of G satisfying $E^2=E$. If

$$(\forall\ a\in H)(aE=Ea) \qquad (1.5)$$

then

$$\mathcal{G}=\{aE \mid a\in H\} \qquad (1.6)$$

is a HX group on G and its unit element just E.

Proof. Take the surjection $f: H \longrightarrow \mathcal{G}$, $a \longmapsto aE$. Noting $f(ab)=(ab)E=$

$(ab)EE=a(bE)E=a(Eb)E=(aE)(bE)=f(a)f(b)$, so $H\sim\mathcal{G}$. Thus $\mathcal{G}$ is a group. Moreover, $f(e)=eE=E$. So E is the unit element of $\mathcal{G}$. Q.E.D.

Semark 1: Since $H\sim\mathcal{G}$, $H/\ker f\cong\mathcal{G}$. This means that $\mathcal{G}$ formed as above must be isomorphic with the quotient group of a certain subgroup of G.

Remark 2: The inverse proposotion of ($E^2=E \Rightarrow E$ is a semigroup) is not true. For example, let G be the additive group of real numbers and take $E=[1,+\infty)$. Then E is a semigroup, but $E+E=[2,+\infty)\neq E$. If $e\in E$ is assumed, then $E^2=E$ iff E is a semigroup of G.

Example 1.1: Let G be the multiplicative group of positive real numbers and H the set of all positive ration numbers, and take $E=[1,+\infty)$. Since they satisfy the conditions in theorem 1.2, $\mathcal{G}=\{aE \mid a\in H\}=\{[a,+\infty) \mid a\in H\}$ is a HX group on G. If we put $f: H \longrightarrow \mathcal{G}$, $a \longmapsto [a,+\infty)$, then $H\cong\mathcal{G}$.

Example 1.2: Let G be the additive group of real numbers and H all integral numbers. The $\mathcal{G}=\{n+E \mid n\in H\}$ is a HX group on G and $H\cong\mathcal{G}$. It is interesting that $0\notin E$ and the elements in $\mathcal{G}$ may form a countable chain:

$$\ldots\ldots(-2,+\infty)\supset(-1,+\infty)\supset E\supset(1,+\infty)\supset(2,+\infty)\quad\ldots\ldots$$

Example 1.3: Let $(G,+,\leqslant)$ be a partial ordering additive group, and write

$$\mathcal{G}=\{[a,b] \mid a,b\in G\},$$

where $[a,b]=\{c\in G \mid a\leqslant c\leqslant b\}$. In $\mathcal{G}$ we define algebraic operation:

$$[a_1,b_1]+[a_2,b_2]=[a_1+a_2,b_1+b_2]$$

It is easy to know that $\mathcal{G}$ is a HX group on G, and $E=[0]$. Moreover the mappings:

$$f:\mathcal{G}\longrightarrow G \qquad\qquad g:\mathcal{G}\longrightarrow G$$

$$[a,b]\longmapsto a \qquad\qquad [a,b]\longmapsto b$$

are surjective no,omorphisms. Clearly, $\ker f=\{[0,b] \mid b\in G\}$, $\ker g=\{[a,0] \mid a\in G\}$. So $\mathcal{G}/\ker f\cong G\cong \ker f\cong \ker g\cong \mathcal{G}/\ker g$.

Theorem 1.2 means that, if we give a semigroup E in G with some condations, a HX group $\mathcal{G}$ can be formed by a subgroup H of G, and E is the unit element of $\mathcal{G}$. Moreover $H\sim\mathcal{G}$. Therefore, we have answered the problem 1 in the affirmative.

2. BASIC THEOREM 2

As the inverse problem of the theorem 1.2, we have

Problem 2: If $\mathcal{G}$ is a HX group on G, whether is there a subgroup H of G such that $\mathcal{G}=\{aE \mid a\in H\}$?

Let $\mathcal{G}$ be a HX group on G. Write

$$G^{*}=\cup\{A \mid A\in\mathcal{G}\} \tag{2.1}$$

$$G^{o}=\{a\in G^{*} \mid a^{-1}\in G^{*}\}$$

PROPOSITION 2.1. 1) G^{*} is a semigroup of G;

2) $G^{o}\neq\emptyset$ iff $e\in G^{o}$;

3) $G^{o}\neq\emptyset$ iff G^{o} is a subgroup of G.

Proof. 1) $a,b\in G^{*}\Longrightarrow(\exists A,B\in\mathcal{G})(a\in A,b\in B)\Longrightarrow ab\in AB\subset G^{*}$.

2) Obvious.

3) $a,b\in G^{o}\Longrightarrow a^{-1},b^{-1}\in G^{*}\Longrightarrow(ab^{-1})^{-1}=ba^{-1}\in G^{*}\Longrightarrow ab^{-1}\in G^{o}\Longrightarrow G^{o}$ is a subgroup of G. Q.E.D.

PROPOSITION 2.2. Let $\mathcal{G}$ be a regular HX group. If $G^{o}\subset H\subset G^{*}$, then H is a subgroup of G $\Longleftrightarrow H=G^{o}$.

Proof. $\Longrightarrow$: If $H\neq G^{o}$, take $a\in H-G^{o}$. Thus $a^{-1}\in H\subset G^{*}$. So $a\in G^{o}$. This is in contradiction with $a\notin G^{o}$.

$\Longleftarrow$: Obvious. Q.E.D.

BASIC THEOREM 1. Let $\mathcal{G}$ be a HX group on G. If E is a subgroup of G, then

1) $\mathcal{G}=\{aE \mid a\in G^{*}\}$;

2) G^{*} is a subgroup of G.

Proof. 1) $\forall A\in\mathcal{G}$, take $a\in A$. We have $aE\subset AE=A$. It can be proved that $aE=A$. If it is not true, then there exists $b\in A-aE$. Then we have $a^{-1}b\notin E$ because $b=ace\in aE$ if $a^{-1}b=c\in E$. For $d\in A^{-1}$ we have da and $db\in A^{-1}A=E$. Thus $a^{-1}b=a^{-1}d^{-1}db=(da)^{-1}(db)\in E$. This is in contradication with $a^{-1}b\notin E$. So $aE=A$. This means that $\mathcal{G}\subset\{aE \mid a\in G^{*}\}$.

Conversely, $\forall a\in G^{*}$, $\exists A\in\mathcal{G}$, such that $a\in A$. So $aE=A\in\mathcal{G}$. Thus $\{aE \mid a\in G^{*}\}\subset\mathcal{G}$.

2) $\forall a\in G^{*}$, $\exists A\in\mathcal{G}$, such that $a\in A$. Noting $e\in E$ and $AA^{-1}=E$, then there

exist $b \in A$, $b^{-1} \in A^{-1}$, such that $bb^{-1}=e$. From $A=bE$ we have $c \in E$ such that $a=bc$. So $a^{-1}=(bc)^{-1}=c^{-1}b^{-1} \in EA^{-1}=A^{-1} \subset G^*$. Q.E.D.

Under the condations of the basic theirem 1, we have the following:

COROLLARY. 1) E is a normal subgroup of G^*;

2) $G^*=G^0$;

3) $\mathcal{G}=G^*/E$.

The following three specific cases important.

1) If G is a periodic group, then $\mathcal{G}=G^*/E$;

2) If E is a finite set, then $\mathcal{G}=G^*/E$;

3) If E is finite group, then $\mathcal{G}=G^*/E$.

It is easy to prove the following theorem:

THEOREM 2.1. Let f be a homomorphism from G to anther group G'. We have

1) If $\mathcal{G}$ is a HX group on G, then

$$\mathcal{G}'=\{f(A) \mid A \in \mathcal{G}\} \tag{2.2}$$

is a HX group on G' and $\mathcal{G} \sim \mathcal{G}'$;

2) Let f be a surjection. If $\mathcal{G}'$ is a HX group on G', then

$$\mathcal{G}=\{f^{-1}(A') \mid A' \in \mathcal{G}'\} \tag{2.3}$$

is a HX group on G and $\mathcal{G} \sim \mathcal{G}'$.

PROPOSITION 2.3. Let $\mathcal{G}$ be a HX group on G, and $B \in 2^G-\{\emptyset\}$ with $B^2=B$. If B satisfies the condation: $(\forall A \in \mathcal{G})(AB=BA)$, then

$$\mathcal{G}_B=\{AB \mid A \in \mathcal{G}\} \tag{2.4}$$

is a HX group on G and $\mathcal{G} \sim \mathcal{G}_B$.

The proof is straight.

Basic theorem 1 answers affirmatively the problem 2. But the condarions are so intense that $\mathcal{G}$ is iniensified as a quotient group which G^* is with respect to E.

In the following, we will reduce the condations and emphasize the structure of regular HX group.

3. BASIC THEOREM 2

DEFINITION 3.1. Let E be a subsemigroup containing the unit element e of G (i.e. a submonoid). E is called a normal subsemigroup of G , if

$$(\forall a \in G)(aE=Ea) \qquad (3.1)$$

If $\mathcal{G}$ is a regular HX group on G, we may guess that E is a regular subsemigroup of a certain subgroup of G.

Let $\mathcal{G}$ be a HX group. For any $A \in \mathcal{G}$, write

$$\overline{A}=\{a \in A \mid a^{-1} \in A^{-1}\}, \qquad (3.2)$$

$\overline{A}$ is called kernel of A. Put

$$\overline{G}=\cup\{\overline{A} \mid A \in \mathcal{G}\} \qquad (3.3)$$

Clearly we have the following facts:

1) $e \in E \Longrightarrow (\forall A \in \mathcal{G})(\overline{A} \neq \emptyset)$; conversely, $(\exists A \in \mathcal{G})(\overline{A} \neq \emptyset) \Longrightarrow e \in E$;

2) $\overline{G} \neq \emptyset$ iff $e \in E$.

BASIC THEOREM 2. If $\mathcal{G}$ is a regular HX group on G, then

1) $\overline{G}$ is a subgroup of G^{0};

2) $\mathcal{G}=\{aE \mid a \in \overline{G}\}$.

Proof. 1) $\forall a,b \in \overline{G}$, $\exists A,B \in \mathcal{G}$, such that $a \in \overline{A}$, $b \in \overline{B}$. Thus $ab^{-1} \in AB^{-1}=C \in \mathcal{G}$. From $(ab^{-1})^{-1}=ba^{-1} \in BA^{-1}=C^{-1}$, we have $ab^{-1} \in \overline{C} \subset \overline{G}$. So $\overline{G}$ is a subgroup of G^{0}.

2) For any $A \in \mathcal{G}$, $\overline{A} \neq \emptyset$ because $\mathcal{G}$ is regular. Taking $a \in \overline{A}$, we can prove that $aE=A$. Clearly $aE \subset AE=A$. In the other respect, $b \in A$ implies $b=eb=(aa^{-1})b=a(a^{-1}b) \in a(A^{-1}A)=aE$. So $A \subset aE$. Thus $A=aE$. This means that $\mathcal{G} \subset \{aE \mid a \in \overline{G}\}$.

Conversely, $\forall a \in \overline{G}$, $\exists A \in \mathcal{G}$, such that $a \in \overline{A}$. Thus $A=aE$. So $\{aE \mid a \in \overline{G}\} \subset \mathcal{G}$.

Q.E.D.

Under the condations of the basic theorem 2, we have the following facts:

1) E is a normal subsemigroup of $\overline{G}$;

2) $\overline{E}$ is a normal subgroup of $\overline{G}$.

Under the condation of $\mathcal{G}$ being regular, the basic theorem 2 answers the problem 2. In the following, we will describe the structure of regular HX group in detail.

DEFINITION 3.2. Let E be a normal subsemigroup of G. The HX group as following:

$$G|E=\{aE \mid a\in G\} \qquad (3.4)$$

is called quasi-quotient group which G is with respect to E.

According to the definition, the basic theorem 2 means, if $\mathcal{G}$ is regular, then $\mathcal{G}$ must be a quasi-quotient group of a certain subgroup of G:

$$\mathcal{G}=\overline{G}|E \qquad (3.5)$$

THEOREM 3.1. If $\mathcal{G}$ is a regular HX group of G, then

$$\overline{G}/\overline{E}\cong\overline{G}|E \qquad (3.6)$$

Proof. Making the mapping $f:\overline{G}\longrightarrow\overline{G}|E,\ a\longmapsto aE$. Clearly f is surjection. So $\overline{G}/\ker f\cong\overline{G}|E$. We can prove that $\ker f=\overline{E}$.

$a\in\ker f \Rightarrow a=ae\in aE=E \Rightarrow (\exists b\in E)(ab=e) \Rightarrow a^{-1}=b\in E \Rightarrow a\in\overline{E}$.

For the other respect, we first note the fact as the following (it is easy to know by the process of the proof of the basic theorem 2):

$$(\forall A\in\mathcal{G})(a\in\overline{A} \Rightarrow aE=A=Ea) \qquad (3.7)$$

Thus $a\in\overline{E} \Rightarrow aE=E \Rightarrow a\in\ker f$. Q.E.D.

Let $\mathcal{G}$ be a HX group, and write

$$\overline{\mathcal{G}}=\{\overline{A} \mid A\in\mathcal{G}\} \qquad (3.8)$$

THEOREM 3.2. If $\mathcal{G}$ is a regular HX group, then

$$\overline{\mathcal{G}}=\overline{G}/\overline{E} \qquad (3.9)$$

Proof. $\forall\ \overline{A}\in\overline{\mathcal{G}}$, taking $a\in\overline{A}$, then $\overline{A}=\overline{aE}$ (noting (3.7)). We prove

$$\overline{aE}=a\overline{E} \qquad (3.10)$$

$x\in a\overline{E} \Rightarrow (\exists h\in\overline{E})(x=ah) \Rightarrow x^{-1}=h^{-1}a^{-1}\in Ea^{-1}=a^{-1}E=(aE)^{-1} \Rightarrow x\in\overline{aE}$.

$x\in\overline{aE} \Rightarrow (\exists h\in E)(x=ah) \Rightarrow h^{-1}=x^{-1}a\in(aE)^{-1}A=A^{-1}A=E \Rightarrow h\in\overline{E} \Rightarrow x=ah\in a\overline{E}$.

This proves (3.10). So $\overline{\mathcal{G}}\subset\overline{G}/\overline{E}$.

Conversely, for any $a\overline{E}\in\overline{G}/\overline{E}$, there exists $\overline{A}\in\overline{\mathcal{G}}$ since $a\in\overline{G}$, such that $a\in\overline{A}$. Thus $\overline{A}=\overline{aE}=a\overline{E}$. So $\overline{G}/\overline{E}\subset\overline{\mathcal{G}}$. Q.E.D.

REFERENCE

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