FUZZY DISTANCE AND LIMIT OF FUZZY NUMBERS

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Abstract

In this paper, we'll introduce two important concepts of fuzzy distance and limit of fuzzy numbers, and give some elementary properties of the fuzzy distance and the limit of fuzzy numbers.

Keywords: Fuzzy number, Fuzzy distance, Fuzzy limit.

Introduction

D. Dubois, H. Prade, R. Goetschel, W. Voxman and J. He etc. [2, 3, 4] have done much work about fuzzy numbers. They have introduced the distance of two fuzzy numbers and given some properties of the limit of the sequence of fuzzy numbers in the distance. In this paper, we'll introduce a new definition of distance of two fuzzy numbers, which is called fuzzy distance. We'll see that the fuzzy distance of two fuzzy numbers is a fuzzy number and is also the extension of the distance of two real-numbers ($|a - b|$), this is to say the fuzzy distance of two real numbers a and b is a real-number $|a - b|$. We'll also introduce the fuzzy limit of the sequence of fuzzy numbers in fuzzy distance, which has almost all the properties similar to those of the limit of the sequence of real-numbers.

The paper is divided into three sections. In Section 1, after recalling some basic definitions and properties of fuzzy numbers, we introduce the relation " $\leq$ " of two fuzzy numbers and define the least upper bound and the greatest lower bound of a set of fuzzy numbers, and give an expression to each of them.

In Section 2, we introduce the fuzzy distance of two fuzzy numbers, which possesses almost every property of the absolute value of real-numbers.

In Section 3, we introduce the limit of the sequence of fuzzy numbers and obtain some results similar to those of the limit of the sequence of real-numbers.

We'll discuss some important theorems of fuzzy numbers in others paper.

1. Basic Definitions and Properties of Fuzzy Number

Let R be the set of all real-numbers and F(R) all fuzzy subsets defined on R [1].

Definition 1.1. Let $\underline{a} \in F(R)$, $\underline{a}$ is called a fuzzy number, if $\underline{a}$ has the properties:

1) $\underline{a}$ is normal, i.e., there exists $x \in R$ such that

$$\underline{a}(x) = 1;$$

2) Whenever $\lambda \in (0, 1]$, then $a_\lambda = \{x;\ \underline{a}(x) \geq \lambda\}$ is a closed interval, denoted by $[a_\lambda^-, a_\lambda^+]$.

Let F*(R) be the set of all fuzzy numbers.

By decompsition theorem of fuzzy set

$$\underline{a} = \bigcup_{\lambda \in [0,1]} \lambda [a_\lambda^-, a_\lambda^+],$$

for every $\underline{a} \in F^*(R)$.

If we define a(x) by

$$a(x) = 1 \quad \text{iff } x = a;$$
$$= 0 \quad \text{iff } x \neq a,$$

for every $a \in R$, then $a \in F^*(R)$ and

$$a = \bigcup_{\lambda \in [0,1]} \lambda [a, a].$$

<u>Definition 1.2.</u> For every $\underline{a}, \underline{b}, \underline{c} \in F^*(R)$, we say that $\underline{c} = \underline{a} + \underline{b}$, if for every $\lambda \in (0, 1]$, $c_\lambda^- = a_\lambda^- + b_\lambda^-$ and $c_\lambda^+ = a_\lambda^+ + b_\lambda^+$. We say that $\underline{c} = \underline{a} - \underline{b}$, if for every $\lambda \in (0, 1]$, $c_\lambda^- = a_\lambda^- - b_\lambda^+$ and $c_\lambda^+ = a_\lambda^+ - b_\lambda^-$.

<u>Definition 1.3.</u> For every $a \in R$ and every $\underline{a} \in F^*(R)$, we define

$$a \cdot \underline{a} = \begin{cases} \bigcup_{\lambda \in [0,1]} \lambda [a \cdot a_\lambda^-, a \cdot a_\lambda^+] & \text{iff } a \geq 0; \\ \bigcup_{\lambda \in [0,1]} \lambda [a \cdot a_\lambda^+, a \cdot a_\lambda^-] & \text{iff } a < 0. \end{cases}$$

Note that the definitions of multiplication, division, maximal and minimal operations of the fuzzy numbers were also introduced by [1] .

<u>Definition 1.4.</u> For $\underline{a}, \underline{b} \in F^*(R)$, we say that $\underline{a} \leq \underline{b}$, if for every $\lambda \in (0, 1]$,

$$a_\lambda^- \leq b_\lambda^- \quad \text{and} \quad a_\lambda^+ \leq b_\lambda^+ .$$

We say that $\underline{a} < \underline{b}$, if $\underline{a} \leq \underline{b}$ and there exists $\lambda \in (0, 1]$ such that

$$a_\lambda^- < b_\lambda^- \qquad \text{or} \qquad a_\lambda^+ < b_\lambda^+ .$$

We say that $\underline{a} = \underline{b}$, if $\underline{a} \leq \underline{b}$ and $\underline{b} \leq \underline{a}$.

Definition 1.5. For every positive real-number M, there exists $\lambda \in (0, 1]$ such that $M \leq a_\lambda^+$ or $a_\lambda^- \leq -M$, then $\underline{a}$ is called fuzzy infinity, write $\underline{\infty}$.

Definition 1.6. A non-empty A is said to be dense, if for every a, b $\in$ A, a < b, there exists c $\in$ A with a < c < b.

Theorem 1.1. F*(R) is dense.

Proof. Let $\underline{a}$, $\underline{b} \in F^*(R)$, $\underline{a} < \underline{b}$, then there exists $\lambda \in (0, 1]$ such that

$$a_\lambda^- < b_\lambda^- \qquad \text{or} \qquad a_\lambda^+ < b_\lambda^+ ,$$

so

$$a_\lambda^- < (a_\lambda^- + b_\lambda^-)/2 < b_\lambda^- \quad \text{or} \quad a_\lambda^+ < (a_\lambda^+ + b_\lambda^+)/2 < b_\lambda^+ .$$

We define

$$\underline{c} = \bigcup_{\lambda \in [0,1]} \lambda \,[(a_\lambda^- + b_\lambda^-)/2 \,,\, (a_\lambda^+ + b_\lambda^+)/2] ,$$

then obviously, $\underline{a} < \underline{c} < \underline{b}$ and $\underline{c} \in F^*(R)$. ∎

Definition 1.7. Let $A \subset F^*(R)$, if there exists $\underline{M} \in F^*(R)$, $\underline{M} \neq \underline{\infty}$, such that

$$\underline{a} \leq \underline{M} \qquad \text{for every } \underline{a} \in A,$$

then A is said to have an upper bound $\underline{M}$. Similarly, if there exists $\underline{m} \in F^*(R)$, $\underline{m} \neq \underline{\infty}$, such that

$$\underline{m} \leq \underline{a} \qquad \text{for every } \underline{a} \in A,$$

then A is said to have a lower bound $\underline{m}$.

A set with both upper and lower bounds is said to be bounded.

Definition 1.8. $\underline{M} \in F^*(R)$ is called the least upper bound of A ($A \subset F^*(R)$), if $\underline{M}$ has the properties:

1) Whenever $\underline{a} \in A$, we have $\underline{a} \leq \underline{M}$;

2) For any $\varepsilon > 0$, there exists at least one $\underline{a} \in A$ such that

$$\underline{M} < \underline{a} + \varepsilon ,$$

we write $\underline{M} = \sup A$.

Similarly, we introduce

Definition 1.9. $\underline{m} \in F^*(R)$ is called the greatest lower bound of A ($A \subset F^*(R)$), if $\underline{m}$ has the properties:

1) Whenever $\underline{a} \in A$, we have $\underline{m} \leq \underline{a}$;

2) For any $\varepsilon > 0$, there exists at least one $\underline{a} \in A$ such that

$$\underline{m} > \underline{a} - \varepsilon ,$$

we write $\underline{m} = \inf A$.

Theorem 1.2. Let $A \subset F^*(R)$, if A has the least upper bound and the greatest lower bound, then

$$\sup A = \bigcup_{\lambda\in[0,1]} \lambda[\sup_{\underline{a}\in A} a_\lambda^- , \sup_{\underline{a}\in A} a_\lambda^+] ; \qquad (1)$$

$$\inf A = \bigcup_{\lambda\in[0,1]} \lambda[\inf_{\underline{a}\in A} a_\lambda^- , \inf_{\underline{a}\in A} a_\lambda^+] . \qquad (2)$$

Proof. For every $\underline{a} \in A$, since $\underline{a} \leq \sup A$, we have

$$a_\lambda^- \leq (\sup A)_\lambda^- \quad \text{and} \quad a_\lambda^+ \leq (\sup A)_\lambda^+,$$

for every $\lambda \in (0 , 1]$. Therefore

$$\sup_{\underline{a}\in A} a_\lambda^- \leq (\sup A)_\lambda^- \quad \text{and} \quad \sup_{\underline{a}\in A} a_\lambda^+ \leq (\sup A)_\lambda^+ ,$$

thus

$$\bigcup_{\lambda\in[0,1]} \lambda [\sup_{\underline{a}\in A} a_\lambda^- , \sup_{\underline{a}\in A} a_\lambda^+] \leq \sup A;$$

On the other hand, for any $\varepsilon > 0$, by using definition 1.8, there exists at least one $\underline{a} \in A$ such that

$$\sup A < \underline{a} + \varepsilon ,$$

hence we have

$$(\sup A)^-_\lambda \leq a^-_\lambda + \varepsilon \quad \text{and} \quad (\sup A)^+_\lambda \leq a^+_\lambda + \varepsilon ,$$

for every $\lambda \in (0,, 1]$. Thus, for every $\lambda \in (0, 1]$,

$$(\sup A)^-_\lambda \leq \sup_{\underline{a} \in A} a^-_\lambda + \varepsilon \quad \text{and} \quad (\sup A)^+_\lambda \leq \sup_{\underline{a} \in A} a^+_\lambda + \varepsilon .$$

Since ε is arbitrary, then

$$(\sup A)^-_\lambda \leq \sup_{\underline{a} \in A} a^-_\lambda \quad \text{and} \quad (\sup A)^+_\lambda \leq \sup_{\underline{a} \in A} a^+_\lambda ,$$

for every $\lambda \in (0, 1]$, that is to say

$$\sup A \leq \bigcup_{\lambda \in [0,1]} \lambda [\sup_{\underline{a} \in A} a^-_\lambda , \sup_{\underline{a} \in A} a^+_\lambda] .$$

It follows that

$$\sup A = \bigcup_{\lambda \in [0,1]} \lambda [\sup_{\underline{a} \in A} a^-_\lambda , \sup_{\underline{a} \in A} a^+_\lambda] ,$$

and this completes the proof of (1).

The proof of (2) is similar. ∎

2. Fuzzy Distance of Fuzzy Numbers and its Properties

Definition 2.1. A fuzzy distance of fuzzy numbers is a function $\underline{\rho}: (F^*(R), F^*(R)) \longrightarrow F^*(R)$ with the properties:

1) $\underline{\rho}(\underline{a}, \underline{b}) \geq 0$, $\underline{a} = \underline{b}$ if and only if $\underline{\rho}(\underline{a}, \underline{b}) = 0$;

2) $\underline{\rho}(\underline{a}, \underline{b}) = \underline{\rho}(\underline{b}, \underline{a})$;

3) Whenever $\underline{c} \in F^*(R)$, we have

$$\underline{\rho}(\underline{a}, \underline{b}) \leq \underline{\rho}(\underline{a}, \underline{c}) + \underline{\rho}(\underline{c}, \underline{b}).$$

If $\underline{\rho}$ is the fuzzy distance of fuzzy numbers, we call $(R, F^*(R), \underline{\rho})$ a fuzzy distance space.

In the following, we introduce a function $\underline{\rho}$, which plays a key role in the **theory of** fuzzy numbers.

We define

$$(*) \qquad \underline{\rho}(\underline{a}, \underline{b}) = \bigcup_{\lambda\in[0,1]} \lambda\,[|a_1^- - b_1^-|\ ,\ \sup_{\lambda\le\eta\le 1} |a_\eta^- - b_\eta^-| \vee |a_\eta^+ - b_\eta^+|],$$

for $\underline{a}, \underline{b} \in F^*(R)$.

Theorem 2.1. $\underline{\rho}(\underline{a}, \underline{b})$ defined by the equality $(*)$ is a fuzzy distance of fuzzy numbers.

Proof. 1), 2) obvious.

3) By using the triangle inequality of real distance $(|a - b|)$, we have

$$|a_\eta^- - b_\eta^-| \le |a_\eta^- - c_\eta^-| + |c_\eta^- - b_\eta^-|$$

and

$$|a_\eta^+ - b_\eta^+| \le |a_\eta^+ - c_\eta^+| + |c_\eta^+ - b_\eta^+|,$$

for every $\eta \in (0, 1]$, therefore

$$|a_\eta^- - b_\eta^-| \le |a_\eta^- - c_\eta^-| \vee |a_\eta^+ - c_\eta^+| + |c_\eta^- - b_\eta^-| \vee |c_\eta^+ - b_\eta^+|$$

and

$$|a_\eta^+ - b_\eta^+| \le |a_\eta^- - c_\eta^-| \vee |a_\eta^+ - c_\eta^+| + |c_\eta^- - b_\eta^-| \vee |c_\eta^+ - b_\eta^+|,$$

for every $\eta \in (0, 1]$, thus

$$|a_\eta^- - b_\eta^-| \vee |a_\eta^+ - b_\eta^+|$$

$$\le |a_\eta^- - c_\eta^-| \vee |a_\eta^+ - c_\eta^+| + |c_\eta^- - b_\eta^-| \vee |c_\eta^+ - b_\eta^+|,$$

for every $\eta \in (0, 1]$, it follows that

$$|a_\eta^- - b_\eta^-| \vee |a_\eta^+ - b_\eta^+|$$

$$\le \sup_{\lambda\le\eta\le 1} |a_\eta^- - c_\eta^-| \vee |a_\eta^+ - c_\eta^+| + \sup_{\lambda\le\eta\le 1} |c_\eta^- - b_\eta^-| \vee |c_\eta^+ - b_\eta^+|,$$

for every $\eta \in [\lambda, 1]$, $\lambda \in (0, 1]$. Therefore we have

$$\sup_{\lambda \le \eta \le 1} |a_\eta^- - b_\eta^-| \vee |a_\eta^+ - b_\eta^+|$$

$$\le \sup_{\lambda \le \eta \le 1} |a_\eta^- - c_\eta^-| \vee |a_\eta^+ - c_\eta^+| + \sup_{\lambda \le \eta \le 1} |c_\eta^- - b_\eta^-| \vee |c_\eta^+ - b_\eta^+|,$$

for every $\lambda \in (0, 1]$. It results that

$$\rho(\underline{a}, \underline{b}) \le \rho(\underline{a}, \underline{c}) + \rho(\underline{c}, \underline{b}).$$ ▮

It is easy to see that, if $\underline{a}$, $\underline{b}$ are real-numbers, then

$$\rho(\underline{a}, \underline{b}) = |a - b|.$$

Theorem 2.2. Whenever $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d} \in F^*(R)$, $a \in R$, we have

1) $\rho(\underline{a} + \underline{b}, \underline{a} + \underline{c}) = \rho(\underline{b}, \underline{c})$;

2) $\rho(\underline{a} - \underline{b}, \underline{a} - \underline{c}) = \rho(\underline{b}, \underline{c})$;

3) $\rho(\underline{b} - \underline{a}, \underline{c} - \underline{a}) = \rho(\underline{b}, \underline{c})$;

4) $\rho(a \cdot \underline{a}, a \cdot \underline{b}) = |a| \cdot \rho(\underline{a}, \underline{b})$;

5) If $\underline{a} \le \underline{b} \le \underline{c}$, then

$$\rho(\underline{a}, \underline{b}) \le \rho(\underline{a}, \underline{c}) \quad \text{and} \quad \rho(\underline{b}, \underline{c}) \le \rho(\underline{a}, \underline{c});$$

6) If $\underline{a} \le \underline{c} \le \underline{b}$ and $\underline{a} \le \underline{d} \le \underline{b}$, then

$$\rho(\underline{c}, \underline{d}) = 2 \cdot \rho(\underline{a}, \underline{b}).$$

Proof. The proof is obvious. ▮

3. Limit of the Sequence of Fuzzy Numbers

<u>Definition 3.1.</u> Let $\{\underline{a}_n\} \subset F^*(R)$, $\underline{a} \in F^*(R)$, $\{\underline{a}_n\}$ is said to converge to $\underline{a}$ in fuzzy distance ρ, denoted by

$$\lim_{n \to \infty} \underline{a}_n = \underline{a} \quad \text{or} \quad \underline{a}_n \longrightarrow \underline{a} \; (n \longrightarrow \infty),$$

if for arbitrary $\varepsilon > 0$, there exists an integer $N > 0$ such that

$$\rho(\underline{a}_n, \underline{a}) < \varepsilon \qquad \text{as } n \ge N.$$

<u>Theorem 3.1.</u> Let $\{\underline{a}_n\} \subset F^*(R)$, $\underline{a} \in F^*(R)$, then $\{\underline{a}_n\}$ converges

to $\underline{a}$ in fuzzy distance $\underline{\rho}$ if and only if $\{a_{n_\lambda}^-\}$, $\{a_{n_\lambda}^+\}$ converge to a_λ^- , a_λ^+ uniformly for every $\lambda \in (0, 1]$ in usual distance of real-numbers.

Proof. Obvious. ∎

Theorem 3.2. Let $\{\underline{a}_n\}$, $\{\underline{b}_n\} \subset F^*(R)$, $\underline{a}$, $\underline{b} \in F^*(R)$, $a \in R$, if

$$\lim_{n\to\infty} \underline{a}_n = \underline{a} \quad \text{and} \quad \lim_{n\to\infty} \underline{b}_n = \underline{b},$$

then

1) $\lim_{n\to\infty} (\underline{a}_n + \underline{b}_n) = \underline{a} + \underline{b}$;

2) $\lim_{n\to\infty} (\underline{a}_n - \underline{b}_n) = \underline{a} - \underline{b}$;

3) $\lim_{n\to\infty} (a\cdot\underline{a}_n) = a\cdot\underline{a}$.

Proof. 1) For arbitrary $\varepsilon > 0$, since $\lim_{n\to\infty} \underline{a}_n = \underline{a}$ and $\lim_{n\to\infty} \underline{b}_n = \underline{b}$, then we can find two integers $N_1 > 0$, $N_2 > 0$ such that

$$\underline{\rho}(\underline{a}_n , \underline{a}) < \tfrac{1}{2}\varepsilon \quad \text{as } n \geq N_1$$

and

$$\underline{\rho}(\underline{b}_n , \underline{b}) < \tfrac{1}{2}\varepsilon \quad \text{as } n \geq N_2.$$

Let $N = \max(N_1 , N_2)$, then we have

$$\underline{\rho}(\underline{a}_n + \underline{b}_n , \underline{a} + \underline{b}) \leq \underline{\rho}(\underline{a}_n + \underline{b}_n , \underline{a}_n + \underline{b}) + \underline{\rho}(\underline{a}_n + \underline{b} , \underline{a} + \underline{b})$$
$$= \underline{\rho}(\underline{b}_n , \underline{b}) + \underline{\rho}(\underline{a}_n , \underline{a}) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

as $n \geq N$. This means that

$$\lim_{n\to\infty} (\underline{a}_n + \underline{b}_n) = \underline{a} + \underline{b}.$$

2) Similarly, we can prove

$$\lim_{n\to\infty} (\underline{a}_n - \underline{b}_n) = \underline{a} - \underline{b}.$$

3) For arbitrary $\varepsilon > 0$, since $\lim_{n\to\infty} \underline{a}_n = \underline{a}$, we can find an integer $N > 0$ such that

$$\underline{\rho}\,(\underline{a}_n\,,\,\underline{a}) < \frac{\varepsilon}{|a| + 1},$$

as $n \geq N$. Therefore

$$\underline{\rho}\,(a\cdot\underline{a}_n\,,\,a\cdot\underline{a}) = |a|\,\underline{\rho}\,(\underline{a}_n\,,\,\underline{a}) < |a|\,\frac{\varepsilon}{|a| + 1} < \varepsilon\;,$$

as $n \geq N$, this shows that

$$\lim_{n\to\infty} (a\cdot\underline{a}_n) = a\cdot\underline{a}.$$

∎

<u>Theorem 3.3.</u> (Limit uniqueness theorem) If $\lim_{n\to\infty} \underline{a}_n = \underline{a}$ and $\lim_{n\to\infty} \underline{a}_n = \underline{b}$, then

$$\underline{a} = \underline{b}.$$

Proof. For any $\varepsilon > 0$, since $\lim_{n\to\infty} \underline{a}_n = \underline{a}$ and $\lim_{n\to\infty} \underline{a}_n = \underline{b}$, then we can find two integers N_1, $N_2 > 0$ such that

$$\underline{\rho}\,(\underline{a}_n\,,\,\underline{a}) < \tfrac{1}{2}\varepsilon \qquad \text{as } n \geq N_1$$

and

$$\underline{\rho}\,(\underline{a}_n\,,\,\underline{b}) < \tfrac{1}{2}\varepsilon \qquad \text{as } n \geq N_2.$$

Let $N = \max\,(N_1\,,\,N_2)$, then

$$0 \leq \underline{\rho}\,(\underline{a}\,,\,\underline{b}) \leq \underline{\rho}\,(\underline{a}_n\,,\,\underline{a}) + \underline{\rho}\,(\underline{a}_n\,,\,\underline{b}) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon\;,$$

as $n \geq N$. The arbitrariness of ε implies that

$$\underline{\rho}\,(\underline{a}\,,\,\underline{b}) = 0,$$

that is

$$\underline{a} = \underline{b}.$$

∎

<u>Theorem 3.4.</u> Let $\{\underline{a}_n\}$, $\{\underline{b}_n\}$, $\{\underline{c}_n\} \subset F^*(R)$, $\underline{a} \in F^*(R)$, if for every n, $\underline{a}_n \leq \underline{b}_n \leq \underline{c}_n$, and $\lim_{n\to\infty} \underline{a}_n = \underline{a}$, $\lim_{n\to\infty} \underline{c}_n = \underline{a}$, then

$$\lim_{n\to\infty} \underline{b}_n = \underline{a}.$$

Proof. For arbitrary $\varepsilon > 0$, since $\lim_{n\to\infty} \underline{a}_n = \underline{a}$, $\lim_{n\to\infty} \underline{c}_n = \underline{a}$, we can find two integers N_1, $N_2 > 0$ such that

$$\underline{\rho}(\underline{a}_n, \underline{a}) < \varepsilon/3 \quad \text{as } n \geq N_1$$

and

$$\underline{\rho}(\underline{c}_n, \underline{a}) < \varepsilon/3 \quad \text{as } n \geq N_2.$$

Since

$$\underline{a}_n \leq \underline{b}_n \leq \underline{c}_n,$$

for every n, then, as $n \geq N = \max(N_1, N_2)$, we have

$$\underline{\rho}(\underline{b}_n, \underline{a}_n) \leq \underline{\rho}(\underline{c}_n, \underline{a}_n) \leq \underline{\rho}(\underline{c}_n, \underline{a}) + \underline{\rho}(\underline{a}, \underline{a}_n)$$
$$< \varepsilon/3 + \varepsilon/3 = (2/3)\varepsilon .$$

Therefore

$$\underline{\rho}(\underline{b}_n, \underline{a}) \leq \underline{\rho}(\underline{b}_n, \underline{a}_n) + \underline{\rho}(\underline{a}_n, \underline{a}) < \frac{2}{3}\varepsilon + \frac{\varepsilon}{3} = \varepsilon,$$

as $n \geq N$, it follows that

$$\lim_{n\to\infty} \underline{b}_n = \underline{a}. \blacksquare$$

Theorem 3.5. (Boundedness theorem) Let $\{\underline{a}_n\} \subset F^*(R)$, $\underline{a} \in F^*(R)$, $\underline{a} \neq \underline{\infty}$, $\underline{a}_n \neq \underline{\infty}$, $n = 1, 2, 3, \cdots \cdots$, if $\lim_{n\to\infty} \underline{a}_n = \underline{a}$, then there exist $\underline{M}$, $\underline{m}$ $(\neq \underline{\infty}) \in F^*(R)$ such that

$$\underline{m} \leq \underline{a}_n \leq \underline{M},$$

for every n.

Proof. Let $\varepsilon = 1$. Since $\lim_{n\to\infty} \underline{a}_n = \underline{a}$, we can find an integer $N > 0$ such that

$$\underline{a} - 1 \leq \underline{a}_n \leq \underline{a} + 1,$$

as $n \geq N$. Let us denote

$$\underline{M} = \bigcup_{\lambda\in[0,1]} \lambda\,[\max_{x\in\{\underline{a}_1,\underline{a}_2,\cdots\underline{a}_N,\underline{a}+1\}} x_\lambda^-,\ \max_{x\in\{\underline{a}_1,\underline{a}_2,\cdots\underline{a}_N,\underline{a}+1\}} x_\lambda^+];$$

$$\underline{m} = \bigcup_{\lambda\in[0,1]} \lambda\,[\min_{x\in\{\underline{a}_1,\underline{a}_2,\cdots\underline{a}_N,\underline{a}-1\}} x_\lambda^-,\ \min_{x\in\{\underline{a}_1,\underline{a}_2,\cdots\underline{a}_N,\underline{a}-1\}} x_\lambda^+],$$

then, obviously, $\underline{M}$, $\underline{m}$ ($\neq \underline{\infty}$) $\in F^*(R)$, and for every n

$$\underline{m} \leq \underline{a}_n \leq \underline{M}.$$ ∎

Theorem 3.6. (Keeping sign property theorem) Let $\{\underline{a}_n\} \subset F^*(R)$, $\underline{a}$, $\underline{b} \in F^*(R)$, $\lim_{n\to\infty} \underline{a}_n = \underline{a}$, if for every n

$$\underline{a}_n \leq \underline{b}\ (\underline{b} \leq \underline{a}_n),$$

then

$$\underline{a} \leq \underline{b}\ (\underline{b} \leq \underline{a}).$$

Proof. Obvious. ∎

Theorem 3.7. Let $\lim_{n\to\infty} \underline{a}_n = \underline{a}$, $\lim_{n\to\infty} \underline{b}_n = \underline{b}$, then

$$\lim_{n\to\infty} \rho\,(\underline{a}_n, \underline{b}_n) = \rho\,(\underline{a}, \underline{b}).$$

Proof. Obvious. ∎

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