ON SUBJECTIVE ORDER RELATIONS BETWEEN FUZZY NUMBERS

by

A. GONZALEZ*, M. A. VILA** and L.M. de CAMPOS*

* Colegio Universitario de Jaén, 23071 JAEN

** Facultad de Ciencias, 18071 GRANADA

Universidad de Granada (Spain)

ABSTRACT

An order relation between fuzzy numbers is defined by means of a subjective approach. This order is defined on a quotient set derived from a previous equality relation and it enables us to define operators max and min for fuzzy numbers with good regularity properties.

KEYWORDS

Fuzzy numbers, order relation, maximum and minimum.

I.- INTRODUCTION.

Fuzzy numbers were introduced in order to modelize imprecise situation involving real numbers. Comparison of fuzzy numbers have been treatted by several authors (Dubois and Prade, Yager, Delgado et al. ..), but the most part of these works do not consider that in imprecise situations every one usually acts by adding some subjective piece of information to the framework of the problem in order to complet the current ill-defined data. Particularly when one is faced with deciding whether an imprecise quantity is great equal or less than another, one introduce a set of subjective elements (depending on the nature of the problem, his wises, request, etc...) needing to be considered in the mathematical model of this decision process.

The present work deals with this last topic. We will define a total order relation between fuzzy numbers which is the basis to formulate max and min operators .

By using the Extension Principle, Dubois and Prade (1978) defined operators to modelize maximum and minimum of fuzzy numbers; but, they are unlike to the classical ones because in most cases, the result of applying them over two (or more) fuzzy numbers is a fuzzy number different of thoses being compared. This is a great difficulty if one want to utilize them in Optimisation or Decision problems involving fuzzy numbers.

We will present here a formulation for the operators max and min without these disadvantages and with some good regularity properties (associative, idempotent etc.). Moreover, they coincide with Dubois and Prade's operators when they are applied to disjoint fuzzy numbers.

Firstly, we will introduce some basic concepts and results as well as the used notation. Next an order relation between fuzzy numbers shall be formulated. This relation introduces decision maker's subjectivity by considering a set of comparison levels with a priority relation. Finally, max and min operators are directly induced from such order relation and we conclude with an analysis of their most relevant properties.

II.- NOTATION AND BASIC DEFINITIONS.

Definition II.1

A fuzzy subset A of the real line $\mathbb{R}$ with membership function $\mu_A(\cdot)$ is said to be a FUZZY REAL NUMBER (f-number) iff:

a) $\forall\alpha\in(0,1]$ the α-level set of A is a convex set

b) $\mu_A(\cdot)$ is an upper semicontinuous function.

c) $supp(A) = \{x\in\mathbb{R} \;/\; \mu_A(x)>0\}$ is a bounded set

d) $\exists a\in\mathbb{R} \;/\; \mu_A(a)=1$.

We will denote by $\mathfrak{D}$ the set f-numbers.

To handle level indices in [0,1], we define A_0 as the closure of supp(A) and so, for any A in $\mathfrak{D}$ and $\forall\alpha\in[0,1]$, A_α is a closed interval of $\mathbb{R}$. We will denote by $\mathfrak{B}$ the set of such intervals.

III.- N-POINT BASED EQUALITY RELATION.

Now, we will give a definition of equality between fuzzy numbers weaker than the one given by Zadeh [7], which supposes the whole equality of membership functions.

Definition III.1

Let consider $\alpha_i \in [0,1]$, i=1...n, such that $\alpha_i \neq \alpha_j$ $\forall i, j$. The relation noted $\langle \alpha_1 \ldots \alpha_n \rangle$ and given by $\forall A, B \in \mathfrak{D}$;

$$A \langle \alpha_1 \ldots \alpha_n \rangle B \iff A_{\alpha_i} = B_{\alpha_i} \quad ; \forall i \in \{1, \ldots n\}$$

Will be called n-point-based.

It is obvious that for any collection $\{\alpha_i \in [0,1], i=1, \ldots n\}$ $\langle \alpha_1 \ldots \alpha_n \rangle$ is an equivalence relation.

Thus we can stablish the corresponding quotient set

$$\mathfrak{D}(\alpha_1 \ldots \alpha_n) = \mathfrak{D} / \langle \alpha_1 \ldots \alpha_n \rangle.$$

Now on this set the mentioned relation is a real equality. In the following we will consider the set $\mathfrak{D}(\alpha_1 \ldots \alpha_n)$ instead of $\mathfrak{D}$, keeping capital letters A, B, ... to denote the classes and their elements simultaneously.

REMARK

The levels $\{\alpha_i\}$ included into definition III.1 are a way to introduce the subjectivity of the decision maker, because they must be chosen. In the same sense, an order relation (standing for an "a priori" arrangement) in the set $\{\alpha_i\}$ will be required below.

For a concret problem the choice of these levels must be done taking into account the particular approach. It must be remarked the number of $\{\alpha_i\}$ is not fixed and so one can use enough levels to guarantee a given precision. Empirical studies indicate n=4 or n=5 are sufficient in most cases. Notice that n=2 assures the equality, in Zadeh's sense, if one use triangular fuzzy numbers

IV.- COMPARISON ON $\mathfrak{D}(\alpha_1 \ldots \alpha_n)$.

This paragraph is devoted to establish a total order relation in $\mathfrak{D}(\alpha_1 \ldots \alpha_n)$. For this purpose, a comparison between real intervals must be previously considered. Actually, any strict order relation defined on $\mathbb{R}^2$ being weakly complete may be useful. Next, we will give the one used in this work.

Definition IV.1

Let $R,S\in\mathfrak{B}$ be such that $R=[r_1,r_2]$ and $S=[s_1,s_2]$ we will say $R\rho S$ iff

$$\{r_2<s_2\} \text{ or } \{r_2=s_2 \text{ and } r_1<s_1\}.$$

The relation ρ is irreflexive, assymetric, transitive and weakly complete in $\mathfrak{B}$. Since it can be interpreted as a lexicographic strict order relation on $\mathbb{R}^2$, using the second component.

Now, on the basis of ρ, we can define an order relation on $\mathfrak{D}(\alpha_1 \ldots \alpha_n)$. Obviously this relation is depending on the set $\{\alpha_i\}$ and it is formulated in the following way.

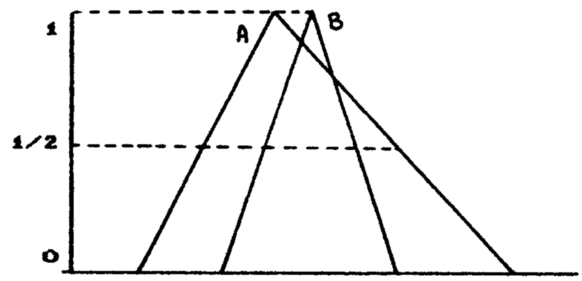
Definition IV.2

Let consider $\{\alpha_i\}$, $i\in I=\{1,2\ldots n\}$ such that $\alpha_i\in[0,1]$ $\forall i$, and its associated $\mathfrak{D}(\alpha_1 \ldots \alpha_n)$. For every $A,B\in\mathfrak{D}(\alpha_1 \ldots \alpha_n)$ we will say

$$A\leq B \text{ iff } \begin{cases} \exists k\in I \text{ such that } A_{\alpha_i}=B_{\alpha_i} \text{ in } 0<i<k \text{ and } A_{\alpha_k}\rho B_{\alpha_k}, \text{ or} \\ \forall i\in I \; A_{\alpha_i}=B_{\alpha_i}. \end{cases}$$

The α_i will be called LEVELS of SUBJECTIVE IMPORTANCE (LSI), and $\leq$ LSI order relation. It is evident the significance of their arrangement, so that a single change on it may lead to different results about the comparison of two fuzzy numbers as can be seen in the following example

EXAMPLE 1



with $\{\alpha_i\}=\{1,1/2,0\}$ we have $A\leq B$ by contrary if we take $\{\alpha_i\}=\{1/2,1,0\}$ then $B\leq A$.

Therefore, as we have said above, the LSI's choice implies an arrangement to be made by the decision maker too.

Now , the properties of $\leq$ are analyzed. It becomes to be a total order relation on $\mathfrak{D}(\alpha_1\ldots\alpha_n)$.

<u>Proposition IV.2</u>

The relation $\leq$ is a total order relation on $\mathfrak{D}(\alpha_1\ldots\alpha_n)$ for every $\{\alpha_i\}$.

Proof

We must prove $\leq$ is reflexive, antisymetric, transitive and total on $\mathfrak{D}(\alpha_1\ldots\alpha_n)$. First property is obvious, we will treat the another ones.

i) $\leq$ is antisymetric.

Let $A,B\in\mathfrak{D}(\alpha_1\ldots\alpha_n)$ by definition

$$A\leq B \leftrightarrow \begin{cases} \exists k\in I \ / \ A_{\alpha_i}=B_{\alpha_i} \ \ 0<i<k \text{ and } A_{\alpha_k}\rho B_{\alpha_k} & (1)\\ \text{or} & \\ \forall i\in I \ \ A_{\alpha_i}=B_{\alpha_i} & (2)\end{cases}$$

$$B\leq A \leftrightarrow \begin{cases} \exists h\in I \ / \ A_{\alpha_i}=B_{\alpha_i} \ \ 0<i<l \text{ and } B_{\alpha_h}\rho A_{\alpha_h} & (3)\\ \text{or} & \\ \forall i\in I \ \ A_{\alpha_i}=B_{\alpha_i} & (4)\end{cases}$$

if we suppose $A\leq B$ and $B\leq A$ it must take into account the following alternative situations may arise:

i.1) (1) and (3) holds. Then for k<h we have $A_{\alpha_k} \rho B_{\alpha_k}$ from (1) and $A_{\alpha_k} = B_{\alpha_k}$ from (3), which is a contradiction because ρ is an strict order relation. Similar conclusions are obtained when h≤k.

i.2) (1) and (4) or (2) and (3) are verified. Both cases lead to contradictions like than i.1)

i.3) (2) and (4) holds. Then by definition of $\mathfrak{D}(\alpha_1 \ldots \alpha_n)$ we conclude A=B.

Therefore A≤B and B≤A implies A=B and so ≤ is antisymetric.

ii) ≤ is transitive. Let consider $A, B, C \in \mathfrak{D}(\alpha_1 \ldots \alpha_n)$. By definition

$$A \le B \leftrightarrow \begin{cases} \exists h \in I \;/\; A_{\alpha_i} = B_{\alpha_i} \;\forall\; i < l \text{ and } A_{\alpha_h} \rho B_{\alpha_h} & (5) \\ \text{or} & \\ B_{\alpha_i} = A_{\alpha_i} \;\forall i \in I & (6) \end{cases}$$

$$B \le C \leftrightarrow \begin{cases} \exists k \in I \;/ B_{\alpha_i} = C_{\alpha_i} \forall\; i < k \text{ and } B_{\alpha_k} \rho C_{\alpha_k} & (7) \\ \text{or} & \\ B_{\alpha_i} = A_{\alpha_i} \;\forall i \in I & (8) \end{cases}$$

Several cases may be considered again.

ii.1) (5) and (8) hold. Obviously B=C and so A≤B. When (6) and (7) are verified then A≤C because A=B . Finally for (6) and (8) we obtain A=B=C.

ii.2) (5) and (7) are verified. In this case, if we take h=min(k,h), then $A_{\alpha_h} \rho C_{\alpha_h}$ and $A_{\alpha_i} = B_{\alpha_i} = C_{\alpha_i}$ $\forall i < h$ and we conclude A≤C. So the relation ≤ is transitive.

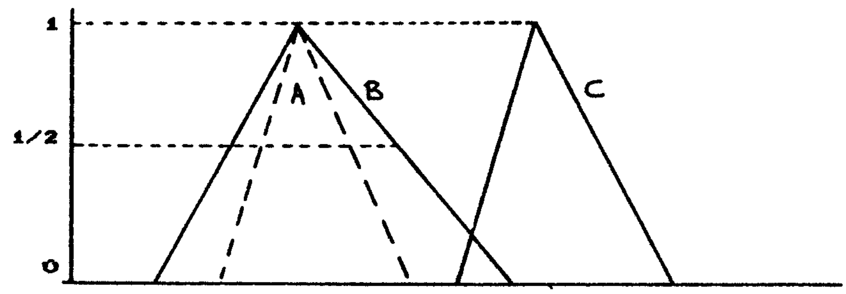
iii) ≤ is total on $\mathfrak{D}(\alpha_1 \ldots \alpha_n)$. Let consider $A, B \in \mathfrak{D}(\alpha_1 \ldots \alpha_n)$ such that n(A≤B). By definition:

$\exists h \in I \;/\; A_{\alpha_h} \ne B_{\alpha_h}$ and if we take $k = \min \{h \in I \;/\; A_{\alpha_h} \ne B_{\alpha_h}\}$

obviously, $\forall i<k$ $A_{\alpha_i}=B_{\alpha_i}$ and $B_{\alpha_k}\rho A_{\alpha_k}$ therefore $B\leq A$ and so $\leq$ is total on $\mathfrak{D}(\alpha_1\ldots\alpha_n)$.□

EXAMPLE 2

The following example shows some applications of LSI-relation between triangular f-numbers with $\{\alpha_i\}=\{1,1/2,0\}$.



$A_1=B_1$ and $A_{1/2}\rho B_{1/2}$ which implies $A\leq B$,
$B_1\rho C_1 \longrightarrow B\leq C$.

V.- <u>APPLICATIONS OF LSI-RELATION. OPERATORS max AND min</u>.

LSI-relation enables us to define operators max and min for f-numbers, with some good properties in order to be used for applications.

<u>Definition V.1</u>.

Let $\{\alpha_i\}$ $i\in I$ be and consider their associated quotient set $\mathfrak{D}(\alpha_1\ldots\alpha_n)$ and the LSI-relation $\leq$. We shall define max and min operators in the following way

$$\forall A,B\in\mathfrak{D}(\alpha_1\ldots\alpha_n)\quad \max(A,B)=\begin{cases} A \text{ if } B\leq A \\ B \text{ if } A\leq B\end{cases} \quad\text{and}\quad \min(A,B)=\begin{cases} B \text{ if } B\leq A \\ A \text{ if } A\leq B\end{cases}$$

The following proposition summarizes several properties of max and min which are evident or can be easily proved.

<u>Proposition V.1</u>

Max and min operators defined above verify

i) $\forall A,B\in\mathfrak{D}(\alpha_1\ldots\alpha_n)$ $\max(A,B)\neq A \Rightarrow \max(A,B)=B$ and $\max(A,B)\neq B \Rightarrow \max(A,B)=A$, (respectively min operator).

ii) $\forall A,B\in\mathfrak{D}(\alpha_1\ldots\alpha_n)$ $\min(A,B) \leq A \leq \max(A,B)$

iii) $\forall A,B\in\mathfrak{D}(\alpha_1\ldots\alpha_n)$ $\min(A,B)=\max(A,B) \Leftrightarrow A=B$

iv) max and min are idempotent operators, i.e. $\max(A,A)=A$ and $\min(A,A)=A$

v) max and min are commutative operators

vi) $\forall A,B\in\mathfrak{D}(\alpha_1\ldots\alpha_n)$ $\max(A,\min(A,B))=\min(A,\max(A,B))=A$

vii) Let * the extension, via Zadeh's principle, of some conmutative operation on $\mathbb{R}$, then $\max(A,B)*\min(A,B)=A*B$.

<u>Proposition V.2</u>

$$\forall A,B,C\in\mathfrak{D}(\alpha_1\ldots\alpha_n) \quad \max(\max(A,B),C)=\max(A,\max(B,C))$$

i.e. max is an associative operator.

Proof:

Let $K=\max(\max(A,B),C)$ and $L=\max(A,\max(B,C))$ be, obviously we can state

$$(K\leq A \wedge K\leq B \wedge K\leq C) \wedge (L\leq A \wedge L\leq B \wedge L\leq B) (9)$$

and by proposition V.1 (i)

$$(K=A \vee K=B \vee K=C) \wedge (L=A \vee L=B \vee L=C)$$

Now we shall suppose $K\neq L$. If we consider $K\leq L$ and $K=A$ then $A\leq L$ and $L\leq A$ so $K=A=L$, which contradicts our hypothesis. Similar considerations can be done in another cases such that, $L\leq K$ $L=B$ etc... All of them lead to contradictions for take $K\neq L$, therefore $K=L$ and the proof is complete. □

REMARK

Obviously this property may be enounced taking min operator instead of max. The proof is similar replacing $K\leq A$ by $A\leq K$, $K\leq B$ by $B\leq K$ etc... in (9).

The associative property allows us to define maximun and minimun for more than two f-numbers in a recursive way.

Definition V.2

Let $A_i \in \mathfrak{D}(\alpha_1 \ldots \alpha_n)$ be, i=1...m we can define

$$\max(A_1 \ldots A_m) = \max(A_1, \max(A_2 \ldots A_m))$$
$$\min(A_1 \ldots A_m) = \min(A_1, \min(A_2 \ldots A_m))$$

Proposition V.3

$\forall A, B, C \in \mathfrak{D}(\alpha_1 \ldots \alpha_n)$

$$\max(A, \min(B, C)) = \min(\max(A, B), \max(A, C))$$

Proof

Let denote

$K = \max(A, \min(B, C))$ and $L = \min(\max(A, B), \max(A, C))$,

obviously

$$(K \leq A \wedge K \leq \min(B, C)) \wedge (\max(A, B) \leq L \wedge \max(A, C) \leq L)$$

and by proposition V.1 (i) $K = A \vee K = \min(B, C)$.

Now, two cases may be differentiated

a) $K=A$, then $A \leq \min(B, C)$ and so
$A \leq B \leq C \Rightarrow L = A$ or $\max(B, C) \leq A \leq \min(B, C) \Rightarrow (B \leq A \leq C) \vee (C \leq A \leq B) \Rightarrow L = A$.

b) $K = \min(B, C)$ then we have
$\min(B, C) \leq A \Rightarrow (\max(A, C) = C \vee \max(A, B) = B) \Rightarrow L = \min(B, C) = K$

The proof is complete.□

REMARK

Like to the proposition V.2, a new formulation may be obtained replacing max by min operator and vice versa. Then we have

$$\min(A, \max(B, C)) = \max(\min(A, B), \min(B, C))$$

and the proof is similar to the above one.

These results can be seen as a way of distributivity.

VI.- CONCLUDING REMARK.

In this paper, we have got establish a way for ranking f-numbers through a total crisp order relation. It enables us to rank any pair of f-numbers obtaining a crisp result.

Moreover, it is a subjective approach, because, a finite set of [0,1]-valued levels must be chosen. It is very important, since it allows different choices to different decision makers.

As a relevant application of our approach we have obtained max and min operators for f-numbers, whose properties are the classical ones for such kind of operators (conmutative, associative, etc,...). Moreover, the application of our operators to two f-numbers gives one of them, unlike other max or min operators.

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