ON THE BASIC FUZZY OPERATIONS

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1. Basic fuzzy operations

Applications of the fuzzy sets theory in several domains, especially in the area of consulting systems, recall the problem of the definition of basic fuzzy operations, namely complementation, intersection and union, see e.g. [4] . These operations are to be defined for each point x of the universal space U as the universal operations on the values of the membership functions, what leads to the analogy with the fuzzy logic connectives. These problems are studied in several papers as [1], [2] or [5] . In what follows we deal with membership values x, y etc. of the unit interval ⟨0 , 1⟩ and the basic fuzzy operations represented by the functions COMP, INT and UN. In the classical fuzzy sets theory following principles are used:

i) the reduction to the non-fuzzy sets operations in the face of binary membership grades

ii) associativity of intersection and union

iii) commutativity of intersection and union

iv) COMP(x) = 1 - x

v) COMP(INT(x,y)) = UN(COMP(x),COMP(y))
COMP(UN(x,y)) = INT(COMP(x),COMP(y))   (the rules of De Morgan)

vi) $INT(x,y) + UN(x,y) = x + y$ (additivity) .

We give three classical examples satisfying i) - vi) :

a) $INT(x,y) = \min(x,y)$ $UN(x,y) = \max(x,y)$

b) $INT(x,y) = x.y$ $UN(x,y) = x + y - x.y$

c) $INT(x,y) = \max(0,x+y-1)$ $UN(x,y) = \min(1,x+y)$.

The work of a rule based consulting system dealing with propositions that may be uncertain consists in the propagation of uncertain knowledge throughout the net of rules according to some combining functions. These functions may be considered as some fuzzy operation functions (see [4]). So e.g. if we take the combining function GLOB (for notation and details see [3]) asuring the calculation of the global weight of a proposition from the particular contributions, we can consider it as an analogy of the fuzzy union, i.e. as a possible type of the function UN (here we deal only with nonnegative weights). In this way it is used the function $GLOB(x,y) = x + y - x.y$, i.e. our type b) of the union function UN, in the consulting systems MYCIN and EMYCIN.

The function UN will be the main subject of our interest. To get some new types (with reasonable properties) for UN, we need to modify or to give up some of the principles i) - vi) .

2. Nonassociative models

Let us preserve all principles i) - vi) with the exception of ii), i.e. of associativity. Then we can use any convex or fuzzy convex combination of the models of type a), b) or c) to get new possible types for UN and INT. As an example we can take the model described in [4] , which is a fuzzy combination of the models a) and b) .

d) $INT(x,y) = (INT_b(x,y), INT_a(x,y), UN_a(x,y)-INT_a(x,y)) =$

$= \min (x,y) -(\min (x,y)).(1 - \max (x,y)).$

$.(\max (x,y) - \min (x,y))$

$UN(x,y) = (UN_b(x,y), UN_a(x,y), UN_a(x,y) - INT_a(x,y)) =$

$= \max (x,y) +(\min (x,y)).(1 - \max (x,y)).$

$.(\max (x,y) - \min (x,y))$

Here $(u,v,z) = z.u + (1-z).v$ is a convex combination.

In this way we can obtain a variety of different types of functions INT and UN. The dissadvantage of this method is the loss of the associativity.

3. Associative models

Most productive generalization seems to be the case, when we preserve the principles i) - iii) and the validity of one of the De Morgan rules.

Let $INT(x,y)$ be any associative commutative representative of the fuzzy intersection increasing in every component, $INT(x,0) = 0$, $INT(x,1) = x$. We modify the principle iv) in the following manner:

iv') COMP is a strictly decreasing transformation of the unit interval $\langle 0 , 1\rangle$ (it follows $COMP(0) = 1$, $COMP(1) = 0$).

Now, we define the function UN by the De Morgan rule,

$UN(x,y) = COMP^{-1}[INT(COMP(x),COMP(y))]$.

It is easy to see, that then UN is an associative commutative representative of the fuzzy union increasing in every component, $UN(x,0) = x$ and $UN(x,1) = 1$.

Conversely, if we are given functions UN and COMP, we define

$$INT(x,y) = COMP^{-1}[UN(COMP(x),COMP(y))] \quad .$$

4. Remark

Now, put $INT(x,y) = x.y$. Let COMP satisfies the modified principle iv*). For $x \neq 1$ we define

$$f(x) = -\ln COMP(x) \quad .$$

Then f is an increasing mapping of $\langle 0 , 1)$ to $\langle 0 , +)$ and we obtain

$$\begin{aligned} UN(x,y) &= COMP^{-1}[INT(COMP(x),COMP(y))] = \\ &= COMP^{-1}(COMP(x).COMP(y)) = \\ &= COMP^{-1}[\exp -(-\ln COMP(x) - \ln COMP(y))] = \\ &= f^{-1}(f(x) + f(y)) \quad . \end{aligned}$$

This result coincides with the results in [3, §2.8] for the combining function GLOB.

5. Power stability

Let again INT and COMP be as in part 4). Then any positive power $COMP^r$, $r>0$, preserve the principle iv*) and it may be regard as a new complementation function $COMP_1$. This new complementation function retains the original definition of UN, as

$$\begin{aligned} UN(x,y) &= COMP^{-1}(COMP(x).COMP(y)) = \\ &= COMP^{-1}(COMP^r(x).COMP^r(y))^{1/r} = \\ &= COMP_1^{-1}(COMP_1(x).COMP_1(y)) \quad . \end{aligned}$$

6. Examples

We give some examples for the function UN following the function COMP (satisfying the principle iv')) and $INT(x,y) = x.y$.

e) $COMP(x) = COMP^{-1}(x) = \frac{1-x}{1+kx}$, $k > -1$

$$UN(x,y) = \frac{x + y + (k-1).x.y}{1 + kx}$$

f) $COMP(x) = \exp(\frac{x}{x-1})$, $COMP^{-1}(x) = \frac{\ln x}{\ln x - 1}$

$UN(x,y) = \frac{x + y - 2x.y}{1 - x.y}$ (here $COMP(1) = 0$, $COMP^{-1}(0) = 1$)

g) $COMP(x) = 1 - x^n$, $n \in N$, $COMP^{-1}(x) = (1-x)^{1/n}$

$UN(x,y) = (x^n + y^n - x^n.y^n)^{1/n}$.

The power stability warrants the same result for $COMP_1(x) = COMP_1^{-1}(x) = (1 - x^n)^{1/n}$.

7. Remarks

I) The inverse $COMP^{-1}$ we can regard as a new complementation function $COMP_2(x) = COMP^{-1}(x)$, $COMP_2^{-1}(x) = COMP(x)$. This exchange may influences the definition of UN. So e.g. in the model g) we have $COMP_2(x) = (1-x)^{1/n}$. With regard to the power stability this case corresponds to the classical $COMP_3(x) = COMP_2^n(x) = 1 - x$, what leads to the model b), i.e. $UN(x,y) = x + y - x.y$.

II) The model f) is the limit model of type e) for $k \downarrow -1$, This is caused by the following facts: the power stability property remains unchanged the definition of UN in the model e) also if we take $COMP_1(x) = COMP^{(1/(k+1))}(x) = (\frac{1-x}{1+kx})^{1/(k+1)}$. For $k \downarrow -1$ we have

$$\lim_{k\downarrow -1} (\frac{1-x}{1+kx})^{1/(k+1)} = \exp(\frac{x}{x-1}) \quad , x \neq -1 .$$

III) The union function UN of the type e) with $k = 1$, i.e. $UN(x,y) = \frac{x + y}{1 + x.y}$ is used as the GLOB function in the consulting system PROSPECTOR .

References

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