QUANTUM LOGICS AND SOFT FUZZY PROBABILITY SPACES

JAROSŁAW PYKACZ

Instytut Matematyki Uniwersytetu Gdańskiego
ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

The aim of my contribution is to compare two notions which have origins in different branches of science but which show remarkable similarities. I hope that such comparison will help to develope both ideas and can indicate direction of further interesting and fruitful investigations.

The first notion - the notion of a soft fuzzy probability space - belongs to the domain of fuzzy set theory and was defined for the first time by Piasecki [1] in the following way:

Definition 1. Let X be a fixed crisp set. A family σ of fuzzy subsets of X is called a **soft fuzzy σ-algebra** if:
(s1) σ contains the constant functions 0 and 1.
(s2) σ does not contain the constant function 1/2.
(s3) σ is closed under standard fuzzy complement, i.e.

$$\text{if } \mu \in \sigma \text{ then } \mu' = 1 - \mu \in \sigma. \tag{1}$$

(s4) σ is closed under countable standard fuzzy union, i.e.

$$\text{if } \mu_1, \mu_2, \ldots \in \sigma \quad \text{then} \quad \bigvee_i \mu_i \in \sigma. \tag{2}$$

Definition 2. Each mapping $p: \sigma \to R^+ \cup \{0\}$ having the following properties:
(p1) for any $\mu \in \sigma$ $p(\mu \vee 1-\mu) = 1$,
(p2) if $\{\mu_i\}$ is a finite or infinite sequence of fuzzy subsets from σ such that $\mu_i \leqslant \mu_j'$ for $i \neq j$ then

$$p(\bigvee_i \mu_i) = \sum_i p(\mu_i) \tag{3}$$

is called a **fuzzy P-measure** on σ and a triplet (X, σ, p) is called a **soft fuzzy probability space.**

Two fuzzy sets μ, ν which fulfill assumptions of the condition (p2) i.e. such that

$$\mu \leqslant 1 - \nu \tag{4}$$

are called **weakly separated sets** by Piasecki in [2]. Any fuzzy set μ such that

$$\mu \leqslant 1 - \mu \tag{5}$$

is in [2] called a **weakly empty set** and any fuzzy set ν such that

$$\nu \geqslant 1 - \nu \tag{6}$$

is called a **weak universum.** As we shall see later all these notions appear as well in the quantum logic theory.

The term "quantum logic" does not have the unique meaning throughout the literature. Its origin lies in the axiomatic approach to quantum mechanics and can be traced back to 1936 to the famous paper of Birkhoff and von Neuman "The logic of quantum mechanics" [3]. The necessity of introducing it was implied by the fact that in quantum physics there exist experimentally verifiable questions which are not simultaneously answerable so the Lindenbaum - Tarski algebra of the quantum theory cannot be a Boolean algebra [4]. Since the Hilbert space formulation of quantum mechanics proved to be very successful and since in this formulation elementary experiments correspond to closed subspaces of a suitable Hilbert space **H**, the lattice **L(H)** of all closed subspaces of a Hilbert space is an object whose various features are chosen by various authors as fundamental properties which define a quantum logic. One of the widest - spread approaches to quantum logic is that of Mackey [5] whose axioms impose on the set of elementary experiments the structure of a partially ordered orthocomplemented σ-orthocomplete orthomodular set with a full set of probability measures. I adopt this point of view but I strongly recommend an excellent reviev book of Beltrametti and Cassinelli [6] to the reader who would like to acquire deeper insight into the physical background of various aspects of this notion.

Let us now collect necessary definitions.

Definition 3. Let L be a partially ordered set ("poset") with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$. The mapping $':L\to L$ is called an **orthocomplementation** iff it has the following properties:

(o1) $a''=a$ for any $a\in L$

(o2) $a\leqslant b$ implies $b'\leqslant a'$

(o3) $a\cup a'=\mathbf{1}$ for any $a\in L$.

The symbol $a\cup a'$ in (o3) denotes the least upper bound of a and a' in L. Conditions (o2) and (o3) imply that the greatest lower bound of a and a', $a\cap a'$, exists in L as well and equals $\mathbf{0}$:

(o3´) $a\cap a'=\mathbf{0}$ for any $a\in L$.

According to the usual definition of orthogonality in orthocomplemented partially ordered sets ("orthoposets") two elements a and b are called **orthogonal** whenever $a\leqslant b'$ or, equivalently by (o1) and (o2), $b\leqslant a'$.

Definition 4. An orthoposet L is called **σ-orthocomplete** iff the least upper bound of every countable sequence of pairwise orthogonal elements exists in L.

Definition 5. An orthoposet L is called **orthomodular** iff for any $a,b\in L$ such that $a\leqslant b$

(om1) $a\cup b'$ exists in L, and

(om2) $b=a\cup(b'\cup a)'$.

Let us notice that if an orthoposet L is σ-othocomplete then the condition (om1) is redundant since $a\leqslant b$ means that a is orthogonal to b' and therefore the least upper bound of a and b' exists in L.

Definition 6. A **probability measure** on a σ-orthocomplete orthoposet L is a map $m:L\to[0,1]$ which satisfies the following conditions:

(m1) $m(\mathbf{1})=1$,

(m2) if $a_1,a_2,\ldots$ is a sequence of pairwise orthogonal elements of L then

$$m(\cup_i a_i)=\sum_i m(a_i)\ .$$

Definition 7. A set $\mathbf{M}$ consisting of probability measures on L is **full** iff $m(a)\leqslant m(b)$ for all $m\in\mathbf{M}$ implies $a\leqslant b$.

It should be stressed that the existence of probability measures on a σ-orthocomplemented orthomodular poset is a nontrivial fact since there exist examples of such posets (even lattices) which do not admit probability measures at all [7].

Definition 8. A partially ordered set **L** satisfying conditions described in definitions 3,4,5 and admitting full set of probability measures is called a **quantum logic**.

The name used above is a traditional one but it is somewhat misleading since we can easily check that any σ-complete Boolean algebra - a structure which is characteristic to classical, not quantum physics - is a quantum logic in the sense of the definition 8. For this reason instead of "quantum logic" I shall often write simply "logic" or, when the physical background of the notion should not be forgotten, "logic of a physical system".

The elements of the logic of a physical system represent the most elementary experiments which can be performed on the physical system under study i.e. experiments with only two possible outcomes. They are called "questions" (possible answers "yes" or "not"), "properties" (possessed or not), "propositions" (true or false), etc. Physicists believe that the whole information about the physical system which we can gain experimentally should be coded in the structure of the logic **L** of the system.

Orthocomplementation in the logic of the physical system is realized as a procedure of passing from any elementary experiment (question) to the experiment performed exactly in the same way with the answer "no" put instead of "yes" and vice versa. The partial order is established as follows:

> $a \leqslant b$ iff, whatever is the state of the system, the probability of obtaining positive answer for the question a is less than or equal to the probability of obtaining positive answer for the question b.

By a state of a system it is meant a collection of attributes which characterize the physical system but which can be different in different situations and may change with time. For this reason the notion of a state of a system is sometimes identified with the prescription for (or procedure of) preparation of a system [8]. Mathematically states are represented by probability measures on **L**. The number m(a) has a physical meaning of the probability of obtaining the positive answer for the question a when the system is in the state m. Therefore we can rewrite the physical interpretation of the partial order in **L** in the form

$$a \leqslant b \quad \text{iff} \quad m(a) \leqslant m(b) \quad \text{for all} \quad m \in \mathbf{S} \tag{7}$$

where **S** denotes the set of all these probability measures on **L** which represent states of a system. We see that the set of all states **S** on a logic **L** is, by the very definition of the partial order in **L**, a full set of probability measures.

States of a physical system, being probability measures on a logic, act as real functionals on **L** but the opposite approach is possible as well: elements of a logic can be treated as real functionals which map **S** into [0,1]. This fact opens the possibility of identifying the elements of the logic **L** with the fuzzy subsets of the set of all states **S**. Such point of view on a quantum logic was proposed in [9]. According to it the number m(a) is interpreted as follows:

> m(a) is the grade of membership of the state m to the subset of **S** which collects all states for which the result of the experiment a is positive.

The idea of treating elements of a logic as functionals on the set of probability measures (without references to fuzzy set theory) was developed by Mączyński [10]. He studied a set of functions L which map a set X into real interval [0,1]. L was equipped with usual pointwise algebraic operations: natural partial order

$$f \leqslant g \quad \text{iff} \quad f(x) \leqslant g(x) \quad \text{for all} \quad x \in X \tag{8}$$

and complementation

$$f' = 1 - f \ . \tag{9}$$

Next, he defined two functions to be **orthogonal** iff $f(x) + g(x) \leqslant 1$ for any $x \in X$ and assumed that in L the following postulate is satisfied:

Orthogonality Postulate. For any sequence of pairwise orthogonal elements of L $f_1, f_2, \ldots$ there is an element $g \in L$ such that $g + f_1 + f_2 + \ldots = 1$.

It occured that the orthogonality postulate is satisfied in L if and only if L has the following properties:

(op1) L contains the constant function 0,

(op2) L is closed under the complementation $f' = 1 - f$,

(op3) L is closed under countable algebraic sums of pairwise orthogonal elements, i.e. if $f_1, f_2, \ldots \in L$, $f_i + f_j \leqslant 1$ then $f_1 + f_2 + \ldots \in L$.

Moreover, Mączyński proved that any set of functions which satisfies the orthogonality postulate is a quantum logic in the sense of the definition 8 and conversly, any quantum logic is isomorphic to such set of functions:

Theorem 1. (Mączyński [10]) Let $L \subseteq [0,1]^X$ satisfy the orthogonality postulate (or, equivalently, properties (op1),(op2) and (op3)). Then L is an orthomodular σ-orthocomplete orthoposet with respect to partial order (8) and complementation (9). Every point $x \in X$ induces a probability measure m_x on $(L, \leqslant, ')$, where $m_x(f) = f(x)$ for all $f \in L$ and the family of all such measures is full.

Conversly, if $(\mathbf{L}, \leqslant, ')$ is an orthomodular σ-orthocomplete orthoposet with a full set $\mathbf{M}$ of probability measures then each $a \in \mathbf{L}$ induces a function $\underline{a}: \mathbf{M} \to [0,1]$ where $\underline{a}(m) = m(a)$ for all $m \in \mathbf{M}$. In the set of all such functions $L = \{\underline{a},\ a \in \mathbf{L}\}$ the orthogonality postulate is satisfied and $(L, \leqslant, ')$ is isomorphic to $(\mathbf{L}, \leqslant, ')$.

Thanks to this theorem any logic of a physical system **L** is isomorphic to a family of a fuzzy subsets of the set of all states **S**. Membership functions of these subsets are defined by the equality $\underline{a}(m) = m(a)$ for every $a \in \mathbf{L}$, $m \in \mathbf{S}$. Because of this isomorphism I shall denote both objects by the same symbol **L** but I shall call **L** either "quantum logic" or "fuzzy quantum logic" if the first or, respectively, the second meaning is given to **L**. Of course we can forget about all physical background and define a fuzzy quantum logic in an abstract way:

Definition 9. Let X be a fixed crisp set. A family L of fuzzy subsets of X will be called a **fuzzy quantum logic** if in L the orthogonality postulate or, equivalently, properties (op1), (op2) and (op3) are satisfied.

When we compare the definition of a fuzzy quantum logic (FQL) with the definition of a soft fuzzy σ-algebra we see that any FQL satisfies conditions (s1) and (s3) of the definition 1. The condition (s2) is by any FQL satisfied as well because of the following theorem:

Theorem 2. A fuzzy quantum logic does not contain any weakly empty set or any weak universum except 0 and 1.

Proof. Let e be any weakly empty set and let u be any weak univwersum in a FQL L. Inequalities (5) and (6) are equivalent to

$$e \leqslant 1/2 \tag{5'}$$

and

$$u \geqslant 1/2 \tag{6'}$$

so $e \leqslant 1-e = e'$ and $u' = 1-u \leqslant u$. Therefore $e \wedge e' = e$ and $u \vee u' = u$ where $\wedge$ and $\vee$ denote, respectively, the greatest lower bound and the least upper bound in L partially ordered by (8). Since by the theorem 1 L is an orthoposet, from the conditions (o3) and (o3′) we obtain $e = 0$ and $u = 1$.

Corollary 1. A fuzzy quantum logic does not contain any constant function except 0 and 1.

This result is from the physical point of view quite natural since if **L** is a fuzzy logic of a physical system and if $a \in$ **L** were a constant function on the set of states **S** then a would give no information about the structure of **S** (which reflects features of a physical system) and therefore would be quite useless. The constant functions 0 and 1 represent trivial experiments with unique outcomes known in advance (for example the constant function 1 can represent an experiment in which we check if a given physical system exists) and they are added to **L** mainly for mathematical convenience.

The fact that a FQL does not contain any constant function except 0 and 1 follows as well from the following theorem:

Theorem 3. If a fuzzy quantum logic contains a function $f \neq 0$ then it does not contain any function kf for $k \in (0,1)$.

Proof. If $k \in (0,1/2)$ then $kf < 1/2$ so $kf \neq 0$ is a weakly empty set and it does not belong to L by the theorem 2.

Let us now suppose that $k \in [1/2,1)$. It was noticed already by Mączyński [10] that in a set of functions L which satisfies the orthogonality postulate if $g \leqslant f$ then $f - g \in L$, therefore if kf belonged to L then $f-kf = (1-k)f$ would belong to L as well. Since $k \in [1/2,1)$, $1-k \in (0,1/2]$ so $(1-k)f \leqslant 1/2$ and $(1-k)f \neq 0$ is a weakly empty set which cannot belong to L by the theorem 2.

Again the fact stated in the theorem 3 is not surprising from the physical point of view. If we have two functions, f and kf, defined on the set of states and representing elementary experiments both of them can be treated as representing measurements of the same physical quantity, the second experimental device being less sensitive. For example let f represents an experiment in which we measure intensity of a beam of particles of a special kind and let the experimental device consists of a filter and a counter placed behind it. Now let replace the counter by a new one which sensity is 80% of the previous one. This new experimental device is represented by the function 0.8f but multiplication of previous results by o.8 does not reflect any change in the beam itself but only a change in a measuring device. It does not reflect any actual feature of a studied physical system and therefore there is no reason to include it in the mathematical description of the system.

We have seen from the proof of the theorem 2 that it is exactly the fact that $f' = 1-f$ is an orthocomplementation in a FQL L which implies that L does not contain any weakly empty set and any weak universum except 0 and 1. Now we shall see that the opposite implication is also true.

Theorem 4. Let F denotes a family of fuzzy subsets of a fixed crisp set X. Let F be closed under standard fuzzy complement (1;9), equipped with the natural partial order (8) and let F contains constant functions 0 and 1. Complementation (1;9) is an orthocomplementation in F if and only if 0 is the only weakly empty set and 1 is the only weak universum in F.

Proof. The "only if" part of the theorem was shown in the proof of the theorem 2.
We shall check now that $f' = 1 - f$ is an orthocomplementation in F. Since conditions (o1) and (o2) of the definition 3 are obviously fulfilled, the condition (o3) is the only one which must be proved.
Let f be any element of F. Of course 1 is an upper bound of f and f'. To show that it is the least upper bound of f and f' let us suppose that $f \leqslant g$ and $f' \leqslant g$. Adding both sides of these inequalities we obtain $1/2 \leqslant g$ so g is a weak universum but since 1 is the only weak universum in F, $f \vee f' = 1$.

As a corollary we obtain the following theorem:

Theorem 5. The standard fuzzy complementation (1;9) is an orthocomplementation in a soft fuzzy σ-algebra $\mathfrak{S}$ equipped with the natural partial order (8) if and only if $\mathfrak{S}$ consists exclusively of crisp sets.

Proof. If a soft fuzzy σ-algebra consists exclusively of crisp sets then it obviously cannot contain any weakly empty set but 0 and any weak universum but 1.
Conversly, let $f \in \mathfrak{S}$. The condition (s4) of the definition 1 implies that $f \vee f' \in \mathfrak{S}$ but $f \vee f' \geqslant 1/2$ so $f \vee f'$ is a weak universum. Since the standard fuzzy complementation (1;9) is an orthocomplementation by the theorem 4 the only weak universum in $\mathfrak{S}$ is 1 so we have $f \vee f' = 1$. This means that for any $x \in X$ $\max(f(x), 1-f(x)) = 1$ so either $f(x) = 1$ or $1-f(x) = 1$, i.e. f is a crisp set.

Since any fuzzy quantum logic is orthocomplemented by (1;9) this theorem shows that crispness of all elements of soft fuzzy σ-algebra is the necessary condition for it to be a fuzzy quantum logic. The next theorem shows that this condition is also sufficient.

Theorem 6. A soft fuzzy σ-algebra $\mathfrak{S}$ is a fuzzy quantum logic if and only if it consists exclusively of crisp sets.

Proof. Since the "only if" part follows directly from the previous theorem and since conditions (s1) and (op1) as well as (s3) and (op2) of the definitions 1 and 9 are identical (of course in (s1) the assumption that $\mathfrak{S}$ contains the constant function 1 is redundant as soon as (s3) is assumed) it suffices to prove (op3) i.e. that $\mathfrak{S}$ is closed under countable algebraic sum of pairwise orthogonal elements.
Let $f_1, f_2, \ldots$ be a sequence of pairwise orthogonal elements. Since all elements in $\mathfrak{S}$ are assumed to be crisp pairwise orthogonality of $f_1, f_2, \ldots$ implies that for any $x \in X$ if $f_i(x)=1$ then $f_j(x)=0$ for all $j \neq i$. Therefore the standard fuzzy union $\bigvee_i f_i$ which belongs to $\mathfrak{S}$ by (s4) in the case of pairwise orthogonal elements coincides with the algebraic sum of these elements and the condition (op3) is proved.

The theorem 6 could not be proved if the roles of a soft fuzzy σ-algebra and a fuzzy quantum logic were exchanged. To show this let us examine the following example:

Example 1. Let $X = [0,1]$ and let $L = \{0, f, f', g, g', 1\}$ where f and g are, respectively, characteristic functions of intervals $[1/3, 2/3]$ and $[0, 1/2]$ restricted to X. It can be easily checked that, besides of pairs which contain 0 (0 is orthogonal to any other element), the only orthogonal pairs are f, f' and

g,g' and that L is a fuzzy quantum logic. L is not a soft fuzzy σ-algebra since for example $f \vee g$ is a characteristic function of the interval $[0,2/3]$ which does not belong to L.

Of course there are numerous examples of fuzzy quantum logics (consisting, because of the theorem 6, exclusively of crisp sets) which are soft fuzzy σ-algebras. For example a family of characteristic functions of any Boolean algebra of crisp subsets of any fixed crisp set is both a FQL and a soft fuzzy σ-algebra. It is caused, as we have seen in the proof of the theorem 5, by the condition (s4) which, when combined with (s3) forces any soft fuzzy σ-algebra to contain together with a non-crisp set f a weak universum $f \vee f' \neq 1$. Let us notice that it was again the condition (s4) which prevented in the example 1 the fuzzy quantum logic L from being a soft fuzzy σ-algebra. Mączyński in the proof of his theorem 1 has shown that in the case of pairwise orthogonal elements of a fuzzy quantum logic the algebraic sum $\sum_i f_i$ coincides with the least upper bound $\bigcup_i f_i$ so the condition (op3) expresses simply the fact that a FQL is a σ-orthocomplete orthoposet. Since in both cases - of a FQL and of a soft fuzzy σ-algebra - the partial order (8) is the same, we see that the difference between fuzzy quantum logics and soft fuzzy σ-algebras is caused mainly by the fact that the standard fuzzy union $\vee$ i.e. a pointwise supremum of membership functions does not coincide with the "global" supremum $\cup$ with respect to the partial order (8) in a FQL $(L,\leq,')$. Of course a FQL as defined by the definition 9 is necessarily closed under suprema of pairwise orthogonal elements only, but even for these authors who assume a quantum logic to be a lattice (cf.|6|) since generally $\vee \neq \cup$, a FQL containing even one non-crisp set would not be a soft fuzzy σ-algebra.

It seems to me that the condition (s4) in the definition 1 was introduced mainly to ensure additivity of fuzzy P-measures. But according to the definition 2 a fuzzy P-measure is assumed to be additive not on arbitrary sequences of fuzzy subsets but only on sequences which consist of pairwise **weakly separated** i.e. according to a FQL terminolgy pairwise **orthogonal** elements. Therefore, according to my oppinion, it would be worthy to weaken the condition (s4) of the definition 1 and assume a soft fuzzy σ-algebra to be closed under countable fuzzy union of sequences of pairwise orthogonal elements only. Such change would not cause many modifications in the whole theory of soft fuzzy σ-algebras and would not cause the notion of a soft fuzzy σ-algebra to be identical with the notion of a fuzzy quantum logic but should make both theories more similar. Particularly, the situation described in the example 1 would be impossible i.e. any FQL consisting exclusively of crisp sets would be a soft fuzzy σ-algebra.

Let me finish with some remarks on probability measures on fuzzy quantum logics and fuzzy P-measures on soft fuzzy σ-algebras.

When we compare definitions 2 and 6 we see that since $1 \vee (1 - 1) = 1$, if a mapping p satisfies the condition (p1) then it satisfies (m1) as well. Since in the case of a FQL the condition (op3) of the definition 9 expresses σ-orthocompleteness of a FQL, we can rewrite condition (m2) of the definition 6 in the form:

(m2') if $f_1, f_2, \ldots$ is a sequence of pairwise orthogonal elements of a fuzzy quantum logic then $m(\sum_i f_i) = \sum_i m(f_i)$.

i.e this condition expresses additivity of a probability measure m on a fuzzy quantum logic with respect to algebraic sums of pairwise orthogonal elements. The condition (p2) of the definition 2 expresses additivity of a fuzzy P-measure as well but with respect to standard fuzzy unions of pairwise orthogonal elements. The standard fuzzy union of a collection of fuzzy subsets coincides with their algebraic sum if and only if the following implication holds

$$f_i(x) \neq 0 \Rightarrow f_j(x) = 0 \text{ for all } j \neq i \text{ and } x \in X$$

so only in such a case , for example if all sets in a pairwise orthogonal sequence are crisp (cf. proof of the theorem 6), the condition (p2) coincides with (m2).

We are ensured by the theorem 1 that any FQL admits a full set of probability measures generated by points in the domain X of L. In the case of soft fuzzy σ-algebras and fuzzy P-measures the situation can be such only if a soft fuzzy σ-algebra consists exclusively of crisp sets i.e. when by the theorem 6 it is actually a fuzzy quantum logic:

Theorem 7. If a soft fuzzy σ-algebra $\mathfrak{S}$ admits a full set of fuzzy P-measures **P** (i.e. if $p(\mu) \leqslant p(\nu)$ for all $p \in \mathbf{P}$ implies $\mu \leqslant \nu$) then:

(i) 1 is the only weak universum in $\mathfrak{S}$,
(ii) 0 is the only weakly empty set in $\mathfrak{S}$,
(iii) $\mathfrak{S}$ consists exclusively of crisp sets.

Proof.
(i) If u is any weak universum in $\mathfrak{S}$ then $p(u)=1$ for any fuzzy P-measure p so we have $p(1)=1 \leqslant 1=p(u)$ for all $p \in \mathbf{P}$. Since the set of fuzzy P-measures **P** is assumed to be full this implies that $1 \leqslant u$ so $u = 1$.
(ii) The proof is analogous to (i).
(iii) Piasecki in [2] has shown that for any $\mu \in \sigma$, $\mu \vee (1-\mu)$ is a weak universum so by (i) $\mu \vee (1-\mu) = 1$ and therefore by the same argument as in the proof of the theorem 5 μ is a crisp set.

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