# On the evolution operator for fuzzy-initiated linear systems

V. N. Bobylev
Computing Center of the Academy of Sciences
Vavilova st. 40, Moscow GSP-1 117967, USSR

> If in arguing about the days of the week we find it convenient to *say* that five and four make two, we may expect to get more paradoxical statements about systems more complicated than the week.
>
> John B. S. Haldane

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## References

[1] V. N. Bobylev, Fuzzy differential equations – A problem letter, BUSEFAL 28 (1985) 56-62.

[2] C. Moler and C. Van Loan, Nineteen dubious ways to compute the exponential of a matrix, SIAM Rev. 20 (1978) 801-836.

## 1. Introduction

For a matrix $A$ and a fuzzy set $\tilde{x}_0$ we denote

$$\tilde{x}_\tau(x_0) = \sum_i \left(\frac{\tau^i}{i!} A^i \tilde{x}_0\right),$$

$$x_\tau(\tilde{x}_0) = \left(\sum_i \frac{\tau^i}{i!} A^i\right) \tilde{x}_0 = e^{A\tau} \tilde{x}_0,$$

$\tau \geqslant 0$ real, $i = 0, 1, 2, \ldots$ integer (the operations will be fixed later). Surprisingly or not, in some instances

$$\tilde{x}_\tau(x_0) \neq x_\tau(\tilde{x}_0).$$

To reveal conditions that ensure the equality is our aim. It is justified by the fact that, firstly, to the fuzzy-valued functions $\tilde{x}_\tau(x_0)$ and $x_\tau(\tilde{x}_0)$ of the variable $\tau$ we assign the sense of evolution operators for the system

$$\frac{d}{d\tau} x = Ax$$

with a fuzzy initial state $\tilde{x}_0$ à la [1]; secondly, on the matrix exponential $e^{A\tau}$ there is an ample literature [2]. Since the difficulties arise even for ordinary, non-fuzzy, sets and since fuzzy sets can be completely described in terms of their level sets, we restrict ourselves to the case of ordinary sets; to emphasize this in what follows we replace $\sim$ by $-$. Presented below should be called the naive approach.

## 2. NOTATION

AS USUAL, BY $R^n$ WE MEAN THE EUCLIDEAN $n$-SPACE (FOR SOME POSITIVE INTEGER $n$). LET

$$x, y \in R^n,$$

$$0 \leq \gamma, \varepsilon \in R^n.$$

LET ALSO $\bar{x}$, $\bar{y}$, $\bar{x}_0$ BE NON-EMPTY COMPACT SUBSETS OF $R^n$, AND $\bar{\gamma}$ A NON-EMPTY COMPACT SUBSET OF THE NON-NEGATIVE REAL NUMBERS. FINALLY, LET $A$ BE A REAL $n$-BY-$n$ MATRIX. EQUIP THE COLLECTION OF ALL $\bar{x}$ WITH THE STANDARD STRUCTURE:

$$\bar{x} = \bar{y} \iff (x \in \bar{x} \iff x \in \bar{y} \ \forall x),$$

$$\bar{x} + \bar{y} = \{x + y : \ x \in \bar{x}, y \in \bar{y}\},$$

$$\gamma \bar{x} = \{\gamma x : \ x \in \bar{x}\},$$

$$\bar{\gamma} x = \{\gamma x : \ \gamma \in \bar{\gamma}\},$$

$$A\bar{x} = \{Ax : \ x \in \bar{x}\},$$

$$\rho(\bar{x}, \bar{y}) = \max\{\max_{x \in \bar{x}} \min_{y \in \bar{y}} |y - x|, \ \max_{y \in \bar{y}} \min_{x \in \bar{x}} |x - y|\}.$$

IT IS IN THIS SENSE THAT THE EQUATION SYMBOLS $\bar{x}_\tau(x_0)$ AND $x_\tau(\bar{x}_0)$ SHOULD BE CONCEIVED.

## 3. Eigen multivectors

**Definition.** $\bar{x}$ is an eigen multivector of $A$ if there exist $\bar{\gamma}$, $x$ and $\gamma$ such that

$$A\bar{x} = \bar{\gamma}x = \gamma\bar{x}.$$

**Lemma.** In this definition it can be supposed that $x \in \bar{x}$, $\gamma \in \bar{\gamma}$ and $Ax = \gamma x$. As a consequence, $e^{A\tau}\bar{x} = e^{\gamma\tau}\bar{x}$.

**Proof.** If $(x \in \bar{x})\,\&\,(\gamma \neq 0)$, with

$$A\bar{x} = \gamma\bar{x} = \bar{\gamma}x,$$

then there exist $y \in \bar{x}$ and $\varepsilon \in \bar{\gamma}$ such that

$$Ax = \gamma y = \varepsilon x.$$

So $Ay = \varepsilon Ax/\gamma = \varepsilon y$ and

$$A\bar{x} = \bar{\gamma}A\frac{x}{\gamma} = \bar{\gamma}y = \frac{\varepsilon}{\gamma}\bar{\gamma}x = \varepsilon\bar{x},$$

hence $Ay = \varepsilon y$ and

$$A\bar{x} = \bar{\gamma}y = \varepsilon\bar{x}.$$

The case $(x \notin \bar{x})\,\&\,(\gamma \neq 0)$ with the subcases $\bar{x} = \{0\}$ and $\bar{x} \neq \{0\}$ reduces to the previous one. The case $\gamma = 0$ is trivial.

## 4. Separability

**Definition.** $\bar{x}$ is $A$-separable if it can be expressed as a sum of eigen multivectors of $A$ or, to put the other way round, as a multisum of eigenvectors of $A$.

**Theorem.** If $\bar{x}_0$ is $A$-separable, then $\bar{x}_\tau(x_0)$ is ($A$-separable and) equal to $x_\tau(\bar{x}_0)$.

**Proof** (by induction). If

$$A\bar{x} = \gamma\bar{x}, \qquad e^{A\tau}\bar{x} = e^{\gamma\tau}\bar{x},$$

then (for $\gamma \geqslant 0$)

$$\bar{x}_\tau(x) = e^{\gamma\tau}\bar{x} = e^{A\tau}\bar{x} = x_\tau(\bar{x}).$$

If now there exist $\bar{x}$ and $\bar{y}$ such that $\bar{x}_0 = \bar{x} + \bar{y}$ and

$$\bar{x}_\tau(x) = x_\tau(\bar{x}),$$

$$\bar{x}_\tau(y) = x_\tau(\bar{y}),$$

then

$$\bar{x}_\tau(x_0) = \bar{x}_\tau(x) + \bar{x}_\tau(y) =$$

$$x_\tau(\bar{x}) + x_\tau(\bar{y}) = x_\tau(\bar{x}_0). \qquad \text{Q.E.D.}$$