FIXED DEGREE THEOREMS OF GENERALIZED FUZZY MAPPING

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This paper obtains some new fixed degree theorems of fuzzy mapping by concept of fixed degree of generalized fuzzy mappings. The results given in this paper improve and extend some results in [1] by Chang Shihsen.

1. Preliminaries

Throughout this paper (X,d) denotes a complete metric space; $H(\cdot,\cdot)$, the Hausdorff metric induced by metric d; CB(X), the collection of all non-empty bounded closed sets of X; $\mathscr{F}(X)$, the collection of all fuzzy sets in X. Let $A\in\mathscr{F}(X)$, $\alpha\in(0,1]$. We write

$$\operatorname{supp} A=\{x\in X:\ A(x)>0\};$$

$$A_\alpha=\{x\in X:\ A(x)=\alpha\};$$

$$\langle A\rangle_\alpha=\{x\in X:\ A(x)\geqslant\alpha\};$$

$$\tilde{A}=\{\xi_\lambda^x;\ \ x\in X,\ A(x)=\lambda\in(0,1]\},$$

where ξ_λ^x is a fuzzy point which takes x as supporting point, λ as value.

Definition 1. Let $A\in\mathscr{F}(X)$, $F:\tilde{A}\longrightarrow\mathscr{F}(X)$ be a mapping, which is called a fuzzy mapping over A, if for each $\xi_\lambda^x\in\tilde{A}$, we have $F(\xi_\lambda^x)\subset A$. We write $F(\xi_\lambda^x)=F_{\xi_\lambda^x}$.

Clearly, if A is an obvious set, then the fuzzy mapping defined above is considered in [1]. The set-valued mapping T: $X \longrightarrow 2^X$ can be taken as a special case of above mentioned fuzzy mapping.

Definition 2. Let $A \in \mathscr{F}(X)$, F be a fuzzy mapping over A, $\xi_\lambda^{x^*} \in A$. If $F_{\xi_\lambda^{x^*}}(x^*)=\alpha$, the $\frac{\alpha}{\lambda}$ is called fixed degree of $\xi_\lambda^{x^*}$ for fuzzy mapping F, we write $D_{fix}(\xi_\lambda^{x^*}, F)=\frac{\alpha}{\lambda}$.

Specifically if $D_{fix}(\xi_\lambda^x, F)=1$, i.e. $F_{\xi_\lambda^x}(x)=\lambda$, then ξ_λ^x is called fixed point of F. If $F_{\xi_\lambda^x}(x)=\max_{u \in X} F_{\xi_\lambda^x}(u)$, then we say that F obtains maximal fixed degree at fuzzy point ξ_λ^x.

Let $A \in \mathscr{F}(X)$, F be a fuzzy mapping over A, if for any $x \in$ supp A, there exists a corresponding $\alpha(x) \in (0,1]$ such that $\{y \in X: F_{\xi_{A(x)}^x}(y)=\alpha(x)\} \in CB(X)$, then we can define a set-valued mapping $\hat{F}$: supp $A \longrightarrow CB(X)$ as follows:

$$\hat{F}(x)=\{y \in X: \ F_{\xi_{A(x)}^x}(y)=\alpha(x)\} \text{ for } \forall x \in \text{supp } A. \qquad (1.1)$$

Cleary, for any $x \in$ supp A, we have $\hat{F}(x) \subset$ supp A, thus for any $y \in \hat{F}(x)$ we have $\xi_{A(y)}^y \in \tilde{A}$. From the definition we can immediately obtain the following result.

Lemma i. Let $A \in \mathscr{F}(X)$, F be a fuzzy mapping over A, $\hat{F}$ be the set-valued mapping defined by F according to (1.1). Then fixed degree of $\xi_{A(x)}^x \in \tilde{A}$ with respect to F is equal to $\frac{\alpha(x)}{A(x)}$ if and only if x is fixed point of the set-valued mapping $\hat{F}$, i.e. $x \in \hat{F}(x)=\{y \in X: \ F_{\xi_{A(x)}^x}(y)=\alpha(x)\}$.

Lemma 2. ([3]). Let $A, B \in CB(X)$, then for any $a \in A$ and $\varepsilon>0$, there exists a point $b \in B$ such that

$$d(a,b) \leqslant H(A,B)+\varepsilon .$$

Lemma 3. Let $\varphi: [0, \infty) \longrightarrow [0, \infty)$ be a increasing function and $\sum_{n=1}^{\infty}\varphi^n(t)<\infty$ for any $t>0$ where $\varphi^n(t)$ is nth composite function of $\varphi(t)$, then we have

(1) $\varphi(t)<t, \quad \forall\, t>0;$ (1.2)

(2) $\varphi(0)=0.$ (1.3)

2. Main results

Theorem 1. Let $A \in \mathscr{F}(X)$, $\langle A\rangle_r \in CB(X)$, $0<r<1$, F,G be two fuzzy mappings over A. If for any $x,y \in$ supp A there are corresponding $\alpha(x)$, $\beta(y) \in [r,1]$ such that $(F_{\xi^x_{A(x)}})_{\alpha(x)}$, $(G_{\xi^y_{A(y)}})_{\beta(y)} \in CB(X)$, and

$$H((F_{\xi^x_{A(x)}})_{\alpha(x)}, (G_{\xi^y_{A(y)}})_{\beta(y)}) \leqslant \Phi(d(x,y), d(x, (F_{\xi^x_{A(x)}})_{\alpha(x)}), d(y, (G_{\xi^y_{A(y)}})_{\beta(y)}), d(x, (G_{\xi^y_{A(y)}})_{\beta(y)}), d(y, (F_{\xi^x_{A(x)}})_{\alpha(x)})), \quad (2.1)$$

where the function $\Phi: [0, \infty)^5 \longrightarrow [0, \infty)$ satisfies the following conditions:

(Φ_1) Φ is a nondecreasing for each variable, and Φ is upper semi-continuous; (2.2)

(Φ_2) $\Phi(t,t,t,at,bt) \leqslant \varphi(t)$, $\forall\, t \geqslant 0$, $a,b=0,1,2;\ a+b=2;$ (2.3)

where $\varphi: [0, \infty) \longrightarrow [0, \infty)$ satisfies the following conditions;

(1) $\varphi(t)$ is strictly increasing over $[0, \infty)$; (2.4)

(2) $\sum_{n=1}^{\infty}\varphi^n(t)<\infty$, $\forall\, t>0.$ (2.5)

Then there exists a fuzzy point $\xi^{x^*}_{A(x^*)} \in \tilde{A}$ such that the common fixed degree of $\xi^{x^*}_{A(x^*)}$ for F and G is equal to

$$\min\left\{\frac{\alpha(x^*)}{A(x^*)},\ \frac{\beta(x^*)}{A(x^*)}\right\}.$$

Proof. Let $\hat{F}$, $\hat{G}$: supp $A \longrightarrow CB(X)$ be two set-valued mappings defined by F and G according to (1.1) respectively. By using Lemma 1, it is sufficient to prove that there exists $x^* \in$ supp A such that $x^* \in (\hat{F}(x^*) \cap \hat{G}(x^*))$.

Taking $x_0 \in$ supp A and $x_1 \in \hat{F}(x_0)$, then $x_1 \in \langle A \rangle_r \subset$ supp A. Let $\alpha > H(\hat{F}(x_0), \hat{G}(x_1))$, by using Lemma 2 there exists $x_2 \in \hat{G}(x_1)$ ($x_2 \in \langle A \rangle_r$), such that $d(x_1, x_2) \leqslant \alpha$.

First we prove that $H(\hat{F}(x_2), \hat{G}(x_1)) < \varphi(\alpha)$. In fact, it follows from (2.1), (2.2), (2.3) and (2.4) that

$$\begin{aligned} H(\hat{G}(x_1), \hat{F}(x_2)) &\leqslant \Phi(d(x_1, x_2), d(x_1, \hat{G}(x_1)), d(x_2, \hat{F}(x_2)), \\ &\qquad d(x_2, \hat{G}(x_1)), d(x_1, \hat{F}(x_2))) \\ &\leqslant \Phi(d(x_1, x_2), d(x_1, x_2)+0, H(\hat{G}(x_1), \hat{F}(x_2)), 0, \\ &\qquad d(x_1, x_2)+H(\hat{G}(x_1), \hat{F}(x_2))) \\ &\leqslant \Phi(d(x_1, x_2), d(x_1, x_2), d(x_1, x_2), 0, \\ &\qquad 2d(x_1, x_2)) \\ &\leqslant \varphi(d(x_1, x_2)) < \varphi(\alpha), \end{aligned}$$

where $d(x_1, x_2) \geqslant H(\hat{G}(x_1), \hat{F}(x_2))$. In fact, if $d(x_1, x_2) < H(\hat{G}(x_1), \hat{F}(x_2))$ then we have

$$H(\hat{G}(x_1), \hat{F}(x_2)) < H(\hat{G}(x_1), \hat{F}(x_2))$$

This is a contradiction.

By using Lemma 2 again, there exists $x_3 \in \hat{F}(x_2)$ $(x_3 \in \langle A \rangle_r)$ such that $d(x_2, x_3) \leq \varphi(\alpha)$. In same way we can prove $H(\hat{F}(x_2), \hat{G}(x_3)) < \varphi(\varphi(\alpha)) = \varphi^2(\alpha)$.

Continuing in this way we can produce a sequence $\{x_n\}_{n=1}^{\infty} \subset \langle A \rangle_r$ such that

$$x_{2n} \in \hat{G}(x_{2n-1}), \qquad n=1,2,\ldots\ldots; \qquad (2.6a)$$

$$x_{2n+1} \in \hat{F}(x_{2n}), \qquad n=0,1,2,\ldots\ldots; \qquad (2.6b)$$

$$d(x_n, x_{n+1}) \leq \varphi^{n-1}(\alpha), \qquad n=2,3,\ldots\ldots . \qquad (2.7)$$

It follows from (2.5) and (2.7) that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $x_n \longrightarrow x^* \in X$. Since $\langle A \rangle_r \in CB(X)$, hence $x^* \in \langle A \rangle_r$, thus $x^* \in$ supp A.

Now we prove that $x^* \in \hat{F}(x^*)$, i.e. $d(x^*, \hat{F}(x^*))=0$. In fact, if $d(x^*, \hat{F}(x^*)) > 0$ we have

$$\begin{aligned} d(x^*, \hat{F}(x^*)) &\leq d(x^*, x_{2n}) + d(x_{2n}, \hat{F}(x^*)) \\ &\leq d(x^*, x_{2n}) + H(\hat{G}(x_{2n-1}), \hat{F}(x^*)) \\ &\leq d(x^*, x_{2n}) + \Phi(d(x_{2n-1}, x^*), d(x_{2n-1}, x_{2n})+0, \\ &\qquad d(x^*, \hat{F}(x^*)), d(x^*, x_{2n})+0, d(x_{2n-1}, x^*)+ \\ &\qquad\qquad d(x^*, \hat{F}(x^*))). \end{aligned}$$

Letting $n \longrightarrow \infty$ on the right side of inequality above, noting the upper semi-continuity of Φ and by using Lemma 3 we have

$$\begin{aligned} d(x^*, \hat{F}(x^*)) &\leq \Phi(0, 0, d(x^*, \hat{F}(x^*)), 0, d(x^*, \hat{F}(x^*))) \\ &\leq \varphi(d(x^*, \hat{F}(x^*))) < d(x^*, \hat{F}(x^*)). \end{aligned}$$

This is a contradiction. From this contradiction it follows that $d(x^*, \hat{F}(x^*))=0$, thus $x^* \in \hat{F}(x^*)$, i.e. $F_{\xi^{x^*}_{A(x^*)}}(x^*)=\alpha(x^*)$. Thus

$$D_{fix}(\xi^{x^*}_{A(x^*)}, F)=\frac{\alpha(x^*)}{A(x^*)}.$$

Similarly we can prove that

$$D_{fix}(\xi^{x^*}_{A(x^*)}, G)=\frac{\beta(x^*)}{A(x^*)}.$$

Therefore

$$D_{fix}(\xi^{x^*}_{A(x^*)}, F\cap G)=\min\left\{\frac{\alpha(x^*)}{A(x^*)}, \frac{\beta(x^*)}{A(x^*)}\right\}.$$

Theorem 2. Let $A\in\mathscr{F}(X)$, $\langle A\rangle_r\in CB(X)$, $0<r<1$, $\{F_i\}_{i=1}^{\infty}$ be a sequence of fuzzy mappings over A. Suppose that for any x, $y\in$ supp A and any positive integers i, j, $i\neq j$ there are corresponding $\alpha_i(x)\in[r, 1]$ such that

$$(F_{i\xi^x_{A(x)}})_{\alpha_i(x)}\in CB(X) \tag{2.8}$$

and

$$\begin{aligned}&H((F_{i\xi^x_{A(x)}})_{\alpha_i(x)}, (F_{j\xi^y_{A(y)}})_{\alpha_j(y)})\\ &\leq\Phi(d(x, y), d(x, (F_{i\xi^x_{A(x)}})_{\alpha_i(x)}), d(y, (F_{j\xi^y_{A(y)}})_{\alpha_j(y)}),\\ &\qquad d(x, (F_{j\xi^y_{A(y)}})_{\alpha_j(y)}), d(y, (F_{i\xi^x_{A(x)}})_{\alpha_i(x)})).\end{aligned} \tag{2.9}$$

Where the function Φ satisfies the conditions (Φ_1), (Φ_2); the function φ in (Φ_2) satisfies the conditions (2.4) and (2.5). Then there exists $\xi^{x^*}_{A(x^*)}\in\tilde{A}$ such that the common fixed degree of $\xi^{x^*}_{A(x^*)}$ for $\{F_i\}_{i=1}^{\infty}$ is equal to $\inf\left\{\frac{\alpha_i(x^*)}{A(x^*)}\right\}$.

The way of proof of this Theorem is analogus to the way of proof of Theorem 1.

The corresponding Theorems in [1] can be extended by same method.

References

[1] Chang Shihsen, Fixed point theorems for fuzzy mappings, Fuzzy sets and systems 17 (1985) 181-187.

[2] Fang Jinxuan, Fixed degree of fuzzy mappings, Science Bulletin 30 (1985) 8:635 (in Chinese).

[3] S.B. Nadler, Jr., Multi-valued contraction mappings, Pacific, J. Math. 30 (1969) 475-488.