ON FUZZY REGULARITY AND SOME WEAKER SEPARATION AXIOMS IN FTS

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1. Introduction

This paper is a summery of our research work. In Section 2, we give a complete comparison of fuzzy regularity concepts presently in use and obtain their characterizations, in Section 3, we introduce and study weaker forms of fuzzy T_1, fuzzy T_2, fuzzy R_o, fuzzy R_1 and fuzzy regular spaces and in Section 4, we note some shortcomings of the fuzzy Hausdorffness concept of Azad [3] and introduce a related concept of fuzzy Hausdorffness, viz. $\underline{\alpha}$-Hausdorffness, which possesses many pleasing properties. We use mainly Chang's topology [4] in this paper. We also use the following facts :

Let x_r be a fuzzy point/singleton in X and α be a fuzzy set in X. Then

(i) $x_r \in \alpha$ iff $r < \alpha(x)$ when $r \in (0,1)$.

(ii) $x_r \underline{\in} \alpha$ iff $r < \alpha(x)$ when $r \in (0,1)$ and $\alpha(x) = 1$

when $r = 1$

(iii) $x_r \leq \alpha$ iff $r \leq \alpha(x)$ and $r \in (0,1]$.

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We also write nhd, in short, for neighbourhood. Let $I = [0,1]$. If $\alpha \in I$, then α shall denote the α-valued constant fuzzy set also. We write 1_A, $D(X)$, G_f and $E(f,g)$ to denote the characteristic function of $A \subseteq X$, diagonal of X, graph of f and equalizer of f, g respectively ($f,g : X \to Y$).

If $d : X \to X \times X$ is the diagonal mapping then

$$d(\alpha) = \alpha \, 1_{D(X)}, \ \forall \ \alpha \in I.$$

2. Fuzzy Regularity Concepts

We begin by giving a list of seven previously introduced fuzzy regularity concepts plus a new one (see def. 2.1(a)) which we additionally propose. The seven previously introduced concepts are due to Hutton and Reilly [6], Adnadjevic [2], Sarkar [13], Ghanim et al. [5], Mahghan and Benchalli [11], Wang [18] and Ali [1]. As all the eight concepts would appear to be qualified for being named "fuzzy regularity concepts", we shall presently distinguish them by labelling them as FR(i)... ....., FR(viii).

Definition 2.1. An fts (X,t) is called

(a) FR(i) iff for each fuzzy point x_r and closed fuzzy set α with $x_r \in co\ \alpha$, $\exists\ \lambda, \mu \in t$ s.t. $x_r \in \lambda$, $\alpha \leq \mu$ and $\lambda \leq co\ \mu$.

(b) FR(ii) iff each $\lambda \in t$ is a union of open fuzzy sets μ_j's s.t. $\bar{\mu}_j \leq \lambda$, for each j [6].

(c) FR(iii) iff for each fuzzy singleton x_r and strong nhd λ of x_r, $\exists$ a strong nhd μ of x_r s.t. $\bar{\mu} \leq \lambda$ [2].

(d) <u>FR(iv)</u> iff for each fuzzy singleton x_r and closed fuzzy set α with $x_r \underline{\in} co\ \alpha$, $\exists\ \lambda, \mu \in t$ s.t. $x_r \underline{\in} \lambda$, $\alpha \leq \mu$ and $\lambda \leq co\ \mu$ [13].

(e) <u>FR(v)</u> iff for each fuzzy singleton x_r and closed fuzzy set α with $x_r \leq co\ \alpha$, $\exists\ \lambda, \mu \in t$ s.t. $x_r \leq \lambda$, $\alpha \leq \mu$ and $\lambda \leq co\ \mu$ [5].

(f) <u>FR(vi)</u> iff for each $x \in X$ and closed fuzzy set α with $\alpha(x) = 0$, $\exists\ \lambda, \mu \in t$ s.t. $\lambda(x) = 1$, $\alpha \leq \mu$ and $\lambda \leq co\ \mu$ [11].

(g) <u>FR(vii)</u> iff for each fuzzy singleton x_r and pseudocrisp closed set α with $x_r \wedge \alpha = 0$, $\alpha \neq 0$, $\exists$ R-nhds λ of x_r and μ of α s.t. $\lambda \vee \mu = 1$ [18].

(h) <u>FR(viii)</u> iff for each fuzzy singleton x_r and closed fuzzy set α with $x_r\ q\ co\ \alpha$, $\exists\ \lambda, \mu \in t$ s.t. $x_r\ q\ \lambda$, $\alpha \leq \mu$ and $\lambda \not{q}\ \mu$ [1].

<u>Theorem</u> 2.1. The following implications hold among the fuzzy regularity concepts FR(i),......FR(viii).

$$FR(v) \Rightarrow FR(iii) \Leftrightarrow FR(iv) \Rightarrow FR(i) \Leftrightarrow FR(ii) \Leftrightarrow FR(viii)$$
$$FR(iv) \Downarrow FR(vi)$$

Furthermore, no other implications exist among FR(i),......., FR(viii).

<u>Theorem</u> 2.2. For an fts (X,t), the following are equivalent :

(a)(i) (X,t) is FR(i) (FR(iv) or FR(v))

(ii) $\forall$ fuzzy point (singleton) x_r and $\lambda \in t$ with $x_r \in \lambda$ ($x_r \underline{\in} \lambda$ or $x_r \leq \lambda$), $\exists\ \mu \in t$ s.t. $x_r \in \mu$ ($x_r \underline{\in} \mu$ or $x_r \leq \mu$) and $\bar{\mu} \leq \lambda$.

(iii) Each fuzzy point (singleton) has a local base of closed nhds.

(iv) Each fuzzy point (singleton) has a local subbase of closed nhds.

(b)(i) (X,t) is FR(vi).

(ii) $\forall\ x \in X$ and $\lambda \in t$ with $\lambda(x) = 1$, $\exists\ \mu \in t$ s.t. $\mu(x) = 1$ and $\bar{\mu} \leq \lambda$.

(iii) Each crisp singleton has a local base of closed nhds.

(iv) Each crisp singleton has a local subbase of closed nhds.

(c)(i) (X,t) is FR(vii)

(ii) $\forall$ fuzzy singleton x_r and pseudocrisp closed set $\alpha \neq 0$ with $x_r \wedge \alpha = 0$, $\exists$ open Q-nhds λ of x_r and μ of α s.t. $\lambda \wedge \mu = 0$.

__Theorem__ 2.3. FR(i) property is initial and hence productive and hereditary.

3. __Some Weaker Separation Axioms__

Following the style of fuzzy T_0-ness introduced and studied by Lowen and Srivastava [10] (which is categorically right), we introduce here the concepts of fuzzy T_1, fuzzy T_2, fuzzy R_0, fuzzy R_1 and fuzzy regular spaces and discuss their properties.

__Definition__ 3.1. An fts (X,t) is called

(a) __FT_0__ iff $\forall\ x, y \in X$, $x \neq y$, $\exists\ \lambda \in t$ s.t. either $\lambda(x) > \lambda(y)$ or $\lambda(y) > \lambda(x)$ [10].

(b) $\underline{FT_1}$ iff $\forall$ x, y $\in$ X, $x \neq y$, $\exists$ $\lambda, \mu \in t$ s.t. $\lambda(x) > \lambda(y)$ and $\mu(y) > \mu(x)$.

(c) $\underline{FT_2}$ iff $\forall$ x, y $\in$ X, $x \neq y$, $\exists$ $\lambda, \mu \in t$ s.t. $\lambda(x) > \lambda(y)$, $\mu(y) > \mu(x)$ and $\lambda \wedge \mu = 0$.

(d) $\underline{FR_o}$ iff $\forall$ x, y $\in$ X, $x \neq y$, whenever $\exists$ $\lambda \in t$ with $\lambda(x) > \lambda(y)$ then $\exists$ $\mu \in t$ with $\mu(y) > \mu(x)$.

(e) $\underline{FR_1}$ iff $\forall$ x, y $\in$ X, $x \neq y$, whenever $\exists$ $\alpha \in t$ with $\alpha(x) \neq \alpha(y)$, then $\exists$ $\lambda, \mu \in t$ s.t. $\lambda(x) > \lambda(y)$, $\mu(y) > \mu(x)$ and $\lambda \wedge \mu = 0$ or equivalently, $\lambda(x) > 0$, $\mu(y) > 0$ and $\lambda \wedge \mu = 0$.

(f) $\underline{FR}$ iff $\forall$ $x \in X$ and closed fuzzy set α with co $\alpha(x) > 0$, $\exists$ $\lambda, \mu \in t$ s.t. $\lambda(x) > 0$, $\alpha \leq \mu$ and $\lambda \leq$ co μ.

It turns out that FT_2-ness is equivalent to fuzzy T_2-ness of Katsaras [7].

Clearly, $FT_2 \Rightarrow FT_1 \Rightarrow FT_o$ and $FR_1 \Rightarrow FR_o$.

Theorem 3.1. FT_1, FR_o, FR_1 and FR properties are good extensions (in the sense of Lowen [8]) of their topological counterparts.

It was shown respectively in [10] and [15] that FT_o and FT_2 properties are good extensions.

Theorem 3.2. Consider the following statements in an fts (X,t):

(i) D(X), the diagonal of X is fuzzy closed in (X x X, t x d), where d is the discrete fuzzy topology on X.

(ii) $\{x\}$, $\forall\ x \in X$, is fuzzy closed in (X,t).
(iii) (X,t) is FT_1.

Then (i) $\Leftrightarrow$ (ii) $\Longrightarrow$ (iii) and (iii) $\not\Longrightarrow$ (ii).

<u>Theorem 3.3</u>. For topologically generated fuzzy topological spaces, the three statements of Th. 3.2 are equivalent.

<u>Theorem 3.4</u>. A topological space (X,T) is compact T_1 iff $(X, \omega(T))$ is fuzzy compact FT_1 (Fuzzy compactness is in the sense of [8].

<u>Theorem 3.5</u> For an fts (X,t), the following statements are equivalent :

(a)(i) (X,t) is FT_1
(ii) (X,t) is FT_o and FR_o.

(b)(i) (X,t) is FT_2
(ii) (X,t) is FT_1 and FR_1
(iii) (X,t) is FT_o and FR_1

(c)(i) (X,t) is FR
(ii) $\forall\ x \in X$ and $\lambda \in t$ with $\lambda(x) > 0$, $\exists\ \mu \in t$ s.t. $\mu(x) > 0$ and $\bar{\mu} \leq \lambda$.

<u>Theorem 3.6</u>. FR_o, FR_1 and FR properties are initial.

<u>Theorem 3.7</u>. FT_1, FR_o, FR_1 and FR properties are productive and hereditary.

4. $\underline{\alpha}$ - Hausdorffness

Several fuzzy Hausdorffness concepts have appeared in the literature so far (a few such significant concepts, together with their comparison, are mentioned in [15] and [17]. In [3], K.K. Azad introduced a fuzzy Hausdorffness concept as follows : An fts (X,t) is called fuzzy Hausdorff : iff $d(\lambda)$ is $t \times t$ - closed $\forall \ \lambda \in I^X$.

Evidently fuzzy Hausdorffness of [3] is also fuzzy Hausdorff in the sense of Srivastava et al. [16](equivalent to the fuzzy Hausdorffness concepts of [9] and [12]); the converse is false.

Considering any discrete topological space as a fuzzy topological space and the 'Fort topological space', we can show respectively that Azad's fuzzy Hausdorffness neither generalizes usual Hausdorffness nor is a good extension of Hausdorffness.

We also observe that the seemingly useful theorem 4.11 in [3], which involves fuzzy Hausdorffness of Azad, survives if his fuzzy Hausdorffness is replaced by the weaker fuzzy Hausdorffness concept given in [16].

To repair these unpleasant aspects of Azad's concept, we introduce here another related fuzzy Hausdorffness concept, viz. $\underline{\alpha}$ - Hausdorffness, which possesses many pleasing properties.

From now onwords, $\alpha \in (0,1]$ and is fixed.

Definition 4.1. An fts (X,t) is said to be $\underline{\alpha}$-Hausdorff iff $\forall \ r, s \in (0,1)$ and $\forall$ distinct $x, y \in X$, $\exists \ \lambda, \mu \in t$ s.t. $\lambda(x) > r$,

$\mu(y) > s$ and $\lambda \wedge \mu \leq 1 - \alpha$.

(A similar name for a different fuzzy Hausdorffness concept has also been used by Rodabangh [14]).

<u>Theorem</u> 4.1. Consider the following statements for an fts (X_1, t_1) :

(i) $\alpha\, 1_{D(X_1)}$ is $t_1 \times t_1$ - closed.

(ii) If (X_2, t_2) is an fts and $f : (X_2, t_2) \rightarrow (X_1, t_1)$ is fuzzy continuous then $\alpha\, 1_{G_f}$ is $t_2 \times t_1$-closed.

(iii) If (X_2, t_2) is an fts and $f, g : (X_2, t_2) \rightarrow (X_1, t_1)$ are fuzzy continuous then $\alpha\, 1_{E(f,g)}$ is t_2 - closed.

(iv) (X_1, t_1) is $\underline{\alpha}$ - Hausdorff.

Then

(a) In Chang's fuzzy topologies

$(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii) \Longrightarrow (iv)$ and $(iv) \not\Longrightarrow (iii)$

(b) In Lowen's fuzzy topologies

$(i) \Longleftrightarrow (ii) \Longleftrightarrow (iii) \Longleftrightarrow (iv)$.

<u>Theorem</u> 4.2. (i) $\underline{\alpha}$ - Hausdorffness is a good extension of its topological counterpart.

(ii) If (X,t) is $\underline{\alpha}$ - Hausdorff then $(X, i(t))$ must be Hausdorff; the converse is not true. (For the converse, consider the counterexample, last but one, of [8]).

<u>Theorem</u> 4.3. $\underline{\alpha}$ - Hausdorffness is productive and hereditary.

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