Abstract Integrals on Fuzzy Sets

(summary)

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This paper is the summary of "Abstract Integrals on Fuzzy Sets", the abstract integrals on the fuzzy sets are introduced, and some properties and transformation theorems of the abstract integrals on the fuzzy sets are discussed, and the Fatou's Lemmas and the Lebesgue's bounded convergence theorems of a sequence of the abstract integrals are proved. The abstract integral studied in the paper is a generalization on the fuzzy set about the integral given in the classical measure theory, and it is different from the Sugeno's fuzzy integral presented in [4].

Throughout this paper, let X be a nonempty set, $\mathfrak{F}(X)$ be the class of all fuzzy subsets of X, $\underset{\sim}{\mathfrak{F}} \subset \mathfrak{F}(X)$ be a fuzzy σ-algebra, and $\underset{\sim}{\mathbb{M}} = \{f;\ f: X \to (-\infty, \infty), \{x; f(x) \geq \alpha\} \in \underset{\sim}{\mathfrak{F}}, \alpha \in [-\infty, \infty]\}$ be the set of all measurable functions on $(X, \underset{\sim}{\mathfrak{F}})$, $\underset{\sim}{\mathbb{M}}^{+} = \{f; f \geq 0, f \in \underset{\sim}{\mathbb{M}}\}$ (cf. [3]), and we make the convention: $0 \cdot \infty = 0$.

1*. Abstract Integrals of Nonnegative Measurable Functions

Definition1.1 Let $\mu: \underset{\sim}{\mathfrak{F}} \to [0, \infty]$ be a fuzzy measure (cf. [3]) it is called σ-additive, if we have $\mu(\bigcup_{n=1}^{\infty} \underset{\sim}{A}_n) = \sum_{n=1}^{\infty} \mu(\underset{\sim}{A}_n)$ whenever $\{\underset{\sim}{A}_n\} \subset \underset{\sim}{\mathfrak{F}}$, and $\underset{\sim}{A}_i \cap \underset{\sim}{A}_j = \Phi$, $i \neq j$.

In the paper, we shall always assume that μ is a σ-additive fuzzy measure.

<u>Definition1.2</u>[1,2] A function f is called simple, if there are some disjoint sets $E_1,\cdots,E_n \in \mathfrak{B}(\underset{\sim}{M})$ (where $\mathfrak{B}(\underset{\sim}{M})$ is the smallest classical σ-algebra such that all functions in $\underset{\sim}{M}$ are measurable on it. Evidently, $\mathfrak{B}(\underset{\sim}{M}) \subset \underset{\sim}{\mathfrak{F}}$) with $\bigcup_{i=1}^{n} E_i = X$, and some real numbers $\alpha_1,\cdots,\alpha_n \in (-\infty, \infty)$, such that $f(x)=\sum_{i=1}^{n}\alpha_i \cdot E_i(x)$, for any $x \in X$, where $E_i(x)=\begin{cases}1 & \text{if } x\in E_i\\ 0 & \text{if } x\notin E_i\end{cases}$.

Evidently, an arbitrary simple function must be measurable.

<u>Proposition1.3</u>[1,2] If $f \in \underset{\sim}{M}^+$, then there exists a sequence $\{f_n\}$ of simple functions in $\underset{\sim}{M}^+$, such that $f_n \nearrow f$.

<u>Definition1.4</u> Let $f = \sum_{i=1}^{n}\alpha_i E_i \in \underset{\sim}{M}^+$ be a simple function, $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$, the integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is defined by $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \triangleq \sum_{i=1}^{n}\alpha_i \cdot \underset{\sim}{\mu}(\underset{\sim}{A} \cap E_i)$.

It follows easily from the additivity of $\underset{\sim}{\mu}$ that $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$ is unambiguously defined.

<u>Definition1.5</u> Let $f \in \underset{\sim}{M}^+$, $\underset{\sim}{A} \in \underset{\sim}{\mathfrak{F}}$. The integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is defined by

$\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \triangleq \sup\{\int_{\underset{\sim}{A}} s du;\ 0 \leq s \leq f,\ s \text{ is a simple function}\}$.

<u>Theorem1.6</u> Let $f \in \underset{\sim}{M}^+$, $\{f_n\} \subset \underset{\sim}{M}^+$ be a sequence of simple functions. If $f_n \nearrow f$, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \lim_{n\to\infty}\int_{\underset{\sim}{A}} f_n d\underset{\sim}{\mu}$.

Evidently, when $\underset{\sim}{\mathfrak{F}}$ is a classical σ-algebra, $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$ is the classical integral given in [1,2].

<u>Theorem1.7</u> Let $f,g \in \underset{\sim}{M}^+$, $\underset{\sim}{A},\underset{\sim}{B} \in \underset{\sim}{\mathfrak{F}}$.

(1) If $\underset{\sim}{\mu}(\underset{\sim}{A}) = 0$, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = 0$;

(2) If $f \leq g$, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \leq \int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$;

(3) If $\underset{\sim}{A}\subset\underset{\sim}{B}$, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \leq \int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$;

(4) $\int_{\underset{\sim}{A}} (f+g) d\underset{\sim}{\mu} = \int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} + \int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$;

(5) $\int_{\underset{\sim}{A}} a f d\underset{\sim}{\mu} = a \int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$, $a\in[0,\infty)$;

(6) $\int_{\underset{\sim}{A}} a d\underset{\sim}{\mu} = a\cdot\underset{\sim}{\mu}(\underset{\sim}{A})$, $a\in[0,\infty)$;

(7) If $\underset{\sim}{A}\cap\underset{\sim}{B}=\phi$, then $\int_{\underset{\sim}{A}\cup\underset{\sim}{B}} f du = \int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} + \int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$;

(8) $\int_{\underset{\sim}{A}} (f\vee g) d\underset{\sim}{\mu} \geq \int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \vee \int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$;

(9) $\int_{\underset{\sim}{A}} (f\wedge g) d\underset{\sim}{\mu} \leq \int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \wedge \int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$;

(10) $\int_{\underset{\sim}{A}\cup\underset{\sim}{B}} f d\underset{\sim}{\mu} \geq \int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \vee \int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$;

(11) $\int_{\underset{\sim}{A}\cap\underset{\sim}{B}} f d\underset{\sim}{\mu} \leq \int_{A} f d\underset{\sim}{\mu} \wedge \int_{\underset{\sim}{B}} f d\underset{\sim}{\mu}$.

2*. Abstract Integrals of Measurable Functions

If $f\in\underset{\sim}{M}$, then we denote $f^+=\max(f,o)$, $f^-=\max(-f,o)$.

Definition2.1 Let $f\in\underset{\sim}{M}$, $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$, if $\int_{\underset{\sim}{A}} f^+ d\underset{\sim}{\mu} < \infty$ or $\int_{\underset{\sim}{A}} f^- d\underset{\sim}{\mu} < \infty$, then we say the integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is existent, and the integral of f on $\underset{\sim}{A}$ with respect to $\underset{\sim}{\mu}$ is defined by $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \triangleq \int_{\underset{\sim}{A}} f^+ d\underset{\sim}{\mu} - \int_{\underset{\sim}{A}} f^- d\underset{\sim}{\mu}$.

If $|\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}| < \infty$, then we say f is integrable.

Evidently, if $\underset{\sim}{\mathfrak{F}}$ is a classical σ-algebra, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$ is the integral presented in [1,2].

Theorem2.2 Let $f,g\in\underset{\sim}{M}$, $\underset{\sim}{A},\underset{\sim}{B}\in\underset{\sim}{\mathfrak{F}}$, and the integrals of f and g on $\underset{\sim}{A}$ and $\underset{\sim}{B}$ with respect to $\underset{\sim}{\mu}$ be existent.

(1) If $\underset{\sim}{\mu}(\underset{\sim}{A})=0$, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = 0$;

(2) If $f\leq g$, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} \leq \int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$;

(3) $\int_{\underset{\sim}{A}} a f d\underset{\sim}{\mu} = a \int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$, $a\in(-\infty,\infty)$;

(4) If g is integrable, then $\int_{\underset{\sim}{A}} (f+g) d\underset{\sim}{\mu}$ is existent, and $\int_{\underset{\sim}{A}} (f+g) d\underset{\sim}{\mu} = \int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} + \int_{\underset{\sim}{A}} g d\underset{\sim}{\mu}$, and therefore, if f and g are integrable, then f+g is integrable;

(5) If $\int_{\underset{\sim}{A} \cup \underset{\sim}{B}} f d\underset{\sim}{\mu}$ is existent and $\underset{\sim}{A} \cap \underset{\sim}{B} = \phi$, then

$$\int_{\underset{\sim}{A} \cup \underset{\sim}{B}} f d\underset{\sim}{\mu} = \int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} + \int_{\underset{\sim}{B}} f d\underset{\sim}{\mu} ;$$

(6) $|\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}| \leq \int_{\underset{\sim}{A}} |f| d\underset{\sim}{\mu}$, and f is integrable if and only if $|f|$ is integrable ;

(7) If $|f| \leq |g|$ and g is integrable, then f is integrable.

Proposition2.3 If $g \leq f$, and g is integrable, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu}$ is existent.

Theorem2.4(Mean Value Theorem) Let $f, g \in \underset{\sim}{M}$, $a, b \in (-\infty, \infty)$, $\underset{\sim}{A} \in \mathfrak{F}$, $f \cdot |g|$ and $|g|$ be integrable, if $a \leq f \leq b$, then there exists $c \in [a, b]$, such that $\int_{\underset{\sim}{A}} f |g| d\underset{\sim}{\mu} = c \int_{\underset{\sim}{A}} |g| d\underset{\sim}{\mu}$.

Corollary2.5 Let $a, b \in (-\infty, \infty)$, $f \in \underset{\sim}{M}$, $\underset{\sim}{A} \in \mathfrak{F}$, and f be integrable, if $a \leq f \leq b$, then there exists $c \in [a, b]$, such that $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = c \cdot \underset{\sim}{\mu}(\underset{\sim}{A})$.

3*. Transformation Theorems

Definition3.1 For any given $\underset{\sim}{A} \in \mathfrak{F}$, we define $\mu^*(E) \triangleq \underset{\sim}{\mu}(\underset{\sim}{A} \cap E)$, for any $E \in \mathfrak{B}(\underset{\sim}{M})$ (cf. Definition1.2), then μ^* is a classical measure on $(X, \mathfrak{B}(\underset{\sim}{M}))$, we call μ^* a measure induced by $\underset{\sim}{\mu}$ and $\underset{\sim}{A}$.

Theorem3.2(Transformation Theorem) If $f \in \underset{\sim}{M}$, $\underset{\sim}{A} \in \mathfrak{F}$, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \int_X f d\mu^*$, where $\int_X f d\mu^*$ is the classical integral of f given in[1,2].

Theorem3.3 (Transformation Theorem) If $\underset{\sim}{A} \in \mathfrak{F}$, $f \in \underset{\sim}{M}$, $S(\underset{\sim}{A}) = \{x;\ \underset{\sim}{A}(x) > 0\} \in \mathfrak{B}(\underset{\sim}{M})$, then $\int_{\underset{\sim}{A}} f d\underset{\sim}{\mu} = \int_{S(\underset{\sim}{A})} f d\mu^*$, where $\int_{S(\underset{\sim}{A})} f d\mu^*$ is the classical integral of f given in[1,2].

4*. Convergence Theorems of Sequence of Measurable Functions

We write $S(\underset{\sim}{A})=\{x;\ \underset{\sim}{A}(x)>0\}$.

Definition4.1 Let $\{f_n, f\}\subset \underset{\sim}{M}$, $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$, $S(\underset{\sim}{A})\in \mathfrak{B}(\underset{\sim}{M})$.

(1) If $\lim_{n\to\infty} f_n(x)= f(x)$ for any $x\in S(\underset{\sim}{A})$, then we say $\{f_n\}$ converges to f everywhere on $\underset{\sim}{A}$, and denote it by $f_n \xrightarrow{e.} f$ on $\underset{\sim}{A}$;

(2) If there exists $E\in\mathfrak{B}(\underset{\sim}{M})$ with $\mu^*(E)=0$, such that $f_n\xrightarrow{e.}f$ on $S(\underset{\sim}{A})\cap E^c$, then we say $\{f_n\}$ converges to f almost everywhere on $\underset{\sim}{A}$, and denote it by $f_n\xrightarrow{a.e.}f$ on $\underset{\sim}{A}$;

(3) If $\underset{\sim}{\mu}(\underset{\sim}{A}\cap\{|f_n-f|\geq\varepsilon\})\to 0$ for any given $\varepsilon>0$, then we say $\{f_n\}$ converges in fuzzy measure $\underset{\sim}{\mu}$ to f on $\underset{\sim}{A}$, and denote it by $f_n\xrightarrow{\underset{\sim}{\mu}}f$ on $\underset{\sim}{A}$;

(4) $\{f_n\}$ is said to mean converge to f on $\underset{\sim}{A}$, if $\int_{\underset{\sim}{A}}|f_n|\,d\underset{\sim}{\mu}<\infty$, $\int_{\underset{\sim}{A}}|f|\,d\underset{\sim}{\mu}<\infty$, and if $\lim_{n\to\infty}\int_{\underset{\sim}{A}}|f_n-f|\,d\underset{\sim}{\mu}=0$.

Theorem4.2 Let $\{f_n, f\}\subset\underset{\sim}{M}$, $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$, $S(\underset{\sim}{A})\in\mathfrak{B}(\underset{\sim}{M})$.

(1) $f_n\xrightarrow{e.}f$ on $\underset{\sim}{A}$ if and only if $\underset{\sim}{A}\subset\{x;\ f_n(x)\to f(x),\ x\in X\}$;

(2) $f_n\xrightarrow{\underset{\sim}{\mu}}f$ on $\underset{\sim}{A}$ if and only if $f_n\xrightarrow{\mu^*}f$ on $S(\underset{\sim}{A})$;

(3) $f_n\xrightarrow{\underset{\sim}{\mu}}f$ on $\underset{\sim}{A}$ if and only if $f_n\xrightarrow{\mu^*}f$ on X .

Theorem4.3 Let $\{f_n, f\}\subset\underset{\sim}{M}$, $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$. If $\{f_n\}$ mean converges to f on $\underset{\sim}{A}$, then $f_n\xrightarrow{\underset{\sim}{\mu}}f$ on $\underset{\sim}{A}$.

Theorem4.4 (Riesz's Theorem) Let $\{f_n, f\}\subset\underset{\sim}{M}$, $\underset{\sim}{A}\in\underset{\sim}{\mathfrak{F}}$, $S(\underset{\sim}{A})\in\mathfrak{B}(\underset{\sim}{M})$. If $f_n\xrightarrow{\underset{\sim}{\mu}}f$ on $\underset{\sim}{A}$, then there exists a subsequence $\{f_{n_i}\}$ of $\{f_n\}$, such that $f_{n_i}\xrightarrow{a.e.}f$ on $\underset{\sim}{A}$.

Theorem4.5 (Lebesgue's Theorem) Let $\{f_n, f\} \subset \underset{\sim}{M}$, $\underset{\sim}{A} \in \mathfrak{F}$, $S(\underset{\sim}{A}) \in \mathfrak{B}(\underset{\sim}{M})$, $\mu(\underset{\sim}{A}) < \infty$. If $f_n \xrightarrow{a.e.} f$ on $\underset{\sim}{A}$, then $f_n \xrightarrow{\mu} f$ on $\underset{\sim}{A}$.

5*. Convergence Theorems of Sequence of Abstract Integrals

Theorem5.1(Monotone Convergence Theorem) Let $f \in \underset{\sim}{M}^+$, $\underset{\sim}{A} \in \mathfrak{F}$, $S(\underset{\sim}{A}) \in \mathfrak{B}(\underset{\sim}{M})$. If $\{f_n\} \subset \underset{\sim}{M}^+$ is an increasing sequence, and if $f_n \xrightarrow{a.e.} f$ on $\underset{\sim}{A}$ (resp. $f_n \xrightarrow{e.} f$ on $\underset{\sim}{A}$), then $\lim_{n\to\infty} \int_{\underset{\sim}{A}} f_n d\mu = \int_{\underset{\sim}{A}} f d\mu$.

Theorem5.2(Monotone Convergence Theorem) Let $f \in \underset{\sim}{M}$, $\underset{\sim}{A} \in \mathfrak{F}$, $S(\underset{\sim}{A}) \in \mathfrak{B}(\underset{\sim}{M})$, $\{f_n\} \subset \underset{\sim}{M}$ be an increasing sequence, $g \in \underset{\sim}{M}$ be a integrable function. If $f \geqslant g$ and $f_n \geqslant g$, $n=1,2,\cdots$, and if $f_n \xrightarrow{a.e.} f$ on $\underset{\sim}{A}$ (resp. $f_n \xrightarrow{e.} f$ on $\underset{\sim}{A}$), then $\lim_{n\to\infty} \int_{\underset{\sim}{A}} f_n d\mu = \int_{\underset{\sim}{A}} f d\mu$.

In the following, we write

$$\underline{\lim}_{n\to\infty} a_n = \sup_n (\inf_{i \geqslant n} a_i) , \quad \overline{\lim}_{n\to\infty} a_n = \inf_n (\sup_{i \geqslant n} a_i) .$$

Theorem5.3(Faton's Lemma) Let $\{g, f_n\} \subset \underset{\sim}{M}$, $\underset{\sim}{A} \in \mathfrak{F}$, $S(\underset{\sim}{A}) \in \mathfrak{B}(\underset{\sim}{M})$. If $g \leqslant f_n$, $n=1,2,\cdots$, and if g is integrable, then

$$\int_{\underset{\sim}{A}} (\underline{\lim}_{n\to\infty} f_n) d\mu \leqslant \underline{\lim}_{n\to\infty} \int_{\underset{\sim}{A}} f_n d\mu .$$

Theorem5.4(Faton's Lemma) Let $\{g, f_n\} \subset \underset{\sim}{M}$, $\underset{\sim}{A} \in \mathfrak{F}$, $S(\underset{\sim}{A}) \in \mathfrak{B}(\underset{\sim}{M})$. If $f_n \leqslant g$, $n=1,2,\cdots$, and if g is integrable, then

$$\int_{\underset{\sim}{A}} (\overline{\lim}_{n\to\infty} f_n) d\mu \geqslant \overline{\lim}_{n\to\infty} \int_{\underset{\sim}{A}} f_n d\mu .$$

Theorem5.5(Lebesgue's Bounded Convergence Theorem) Let $\{f_n, f\} \subset \underset{\sim}{M}$, $\underset{\sim}{A} \in \mathfrak{F}$, $S(\underset{\sim}{A}) \in \mathfrak{B}(\underset{\sim}{M})$, and $g \in \underset{\sim}{M}$ be integrable. If $|f_n| \leqslant |g|$, a.e.($\mu^*(\{|f_n| > |g|\}) = 0$), and if $f_n \xrightarrow{a.e.} f$ on $\underset{\sim}{A}$, then f is integrable, and $\lim_{n\to\infty} \int_{\underset{\sim}{A}} |f_n - f| d\mu = 0$, and therefore,

$\lim_{n\to\infty}\int_A f_n d\mu = \int_A f d\mu$.

Theorem5.6 (Lebesgue's Bounded Convergence Theorem) Let $\{f_n, f\} \subset M$, $A \in \mathcal{F}$, and $g \in M$ be integrable. If $|f_n| \leq |g|$ a.e. , and if $f_n \xrightarrow{\mu} f$ on A , then f is integrable, and $\lim_{n\to\infty}\int_A |f_n - f| d\mu = 0$, and therefore, $\lim_{n\to\infty}\int_A f_n d\mu = \int_A f d\mu$.

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