CALCULUS OF FUZZY SETS III *)

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Summary: A general view on fuzzy relation composition is presented with a list of sufficient assumptions for its usual properties.

Operations on fuzzy relations. Let L be a bounded poset with a binary operation $* : L^2 \to L$, and let $P, Q, X, Y, Z \neq \emptyset$ denote arbitrary sets.

DEFINITION 1 (Zadeh [7], Goguen [5]). Any L-fuzzy set in the Cartesian product $X \times Y$ is called an *L-fuzzy relation* from X to Y. The set of all such fuzzy relations is denoted by $L(X,Y)$.

A fuzzy relation $R \in L(X,Y)$ has its *converse* $R^{-1} \in L(Y,X)$ such that

$$R^{-1}(y,x) = R(x,y) \quad \text{for} \quad x \in X,\ y \in Y.$$

Any $R \in L(X,X)$ is called an L-fuzzy relation on X. In particular, the *identity* fuzzy relation on X is $I_X \in L(X,X)$ such that

$$I_X(x,y) = \begin{cases} 1 & \text{for } x = y \\ 0 & \text{for } x \neq y \end{cases}, \quad x, y \in X. \tag{1}$$

As in the case of fuzzy sets $L(X,Y)$ is ordered by the relation

$$P \leqslant Q \iff (P(x,y) \leqslant Q(x,y) \quad \text{for } x \in X,\ y \in Y), \tag{2}$$

*) It is a part of Chapter III from [3].

and has the extended operation

(3) $(P * Q)(x,y) = P(x,y) * Q(x,y)$ for $x \in X$, $y \in Y$,

where $P, Q \in L(X,Y)$. Using certain assumptions on operation $*$ this can be considered as a union or an intersection of fuzzy relations *).

The main operation on fuzzy relations is their *composition*. For the general case a set operation $\phi : 2^L \to L$ is necessary. It can be extended by the formula

(4) $\phi(U)(x,y) = \phi(\{R(x,y) | R \in U\})$ for $U \subset L(X,Y)$, $x \in X$, $y \in Y$

and different operations ϕ can be distinguished by their arguments.

DEFINITION 2 **). Let $R \in L(X,Y)$, $S \in L(Y,Z)$. The ϕ-$*$ *composition* of R and S is the fuzzy relation $RS \in L(X,Z)$ such that

(5) $(RS)(x,z) = \phi(\{R(x,y) * S(y.z) | y \in Y\})$ for $x \in X$, $z \in Z$.

The ϕ-$*$ n-th *power* of fuzzy relation $R \in L(X,X)$ is a relation $R^n \in L(X,X)$, where

(6) $R^1 = R$, $R^{n+1} = RR^n$ for $n \in \mathbb{N}$.

The ϕ-$*$ *closure* of fuzzy relation $R \in L(X,X)$ is a relation $\bar{R} \in L(X,X)$ such that

(7) $\bar{R} = \phi(\{R^n | n \in \mathbb{N}\})$

with pointwise extension (4) of ϕ.

Properties of the above defined operations depends on the particular assumptions on ϕ and $*$. The main assumptions of this paper has the form:

HYPOTHESIS 1. Operation $*$ is associative and distributive over ϕ, i.e. fulfils

$$a * \phi(\{x_p | p \in P\}) = \phi(\{a * x_p | p \in P\}),$$

$$\phi(\{x_p | p \in P\}) * a = \phi(\{x_p * a | p \in P\})$$

for $a \in L$, $x_p \in L$, $p \in P$.

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*) Cf. Drewniak [2].

**) Cf. Zadeh [8].

Operation ϕ fulfils the law of generalized associativity

(8) $\phi(\{x_{pq} \mid p\in P,\ q\in Q\}) = \phi(\{\phi(\{x_{pq} \mid p\in P\}) \mid q\in Q\})$

for $p \in P$, $q \in Q$, $x_{pq} \in L$,

HYPOTHESIS 2. Operation $*$ has null 0 and identity 1.

HYPOTHESIS 3. Operation ϕ is isotone and $\phi(\{1\}) = 1$.

THEOREM 1. Assume Hypothesis 1. Then

a) the composition (5) is associative.

b) the composition (5) is distributive over ϕ, i.e.

(9) $\phi(\{R_p \mid p\in P\})\phi(\{S_q \mid q\in Q\}) = \phi(\{R_pS_q \mid p\in P,\ q\in Q\})$

for $R_p \in L(X,Y)$, $S_q \in L(Y,Z)$, $p \in P$, $q \in Q$, with pointwise extension (4) of ϕ.

c) the closure (7) fulfils

(10) $(\bar{R})^2 = R\bar{R} = \bar{R}R$, $\bar{\bar{R}} = \bar{R}$ for $R \in L(X,X)$.

d) if we add Hypothesis 3 then

(11) $R \leq \bar{R}$, $(\bar{R})^n \leq \bar{R}$ for $R \in L(X,X)$, $n \in \mathbf{N}$.

e) if we add Hypotheses 2 and 3 then $L(X,X)$ with composition (5) is a semigroup with identity I_X and null $0_{X\times X}$.

P r o o f. To simplify our notation let $R, S, T \in L(X,X)$. We ask whether

(12) $R(ST) = (RS)T$.

For arbitrary $x, u \in X$ consider left and right sides of (12). Using distributivity and generalized associativity we get

$$[R(ST)](x,u) = \phi(\{R(x,y) * \phi(\{S(y,z) * T(z,u) \mid z\in X\}) \mid y\in X\}) =$$

$$= \phi(\{\phi(\{R(x,y) * (S(y,z) * T(z,u)) \mid z\in X\}) \mid y\in X\}) =$$

$$= \phi(\{R(x,y) * (S(y,z) * T(z,u)) \mid y, z \in X\}),$$

$$[(RS)T](x,u) = \phi(\{\phi(\{R(x,y) * S(y,z) \mid y\in X\}) * T(z,u) \mid z\in X\}) =$$

$$= \phi(\{\phi(\{(R(x,y) * S(y,z)) * T(z,u) \mid y\in X\}) \mid z\in X\}) =$$

$$= \phi(\{(R(x,y) * S(y,z)) * T(z,u)) \mid y, z \in X\}).$$

Thus, by associativity of $*$, the equality (12) is proved.

Now let consider left and right sides of (9) for $x \in X$, $z \in Z$. Using Hypothesis 1 we get

$$[\phi(\{R_p|\ p\in P\})\phi(\{S_q|\ q\in Q\})](x,z) =$$

$$= \phi(\{\phi(\{R_p(x,y)|\ p\in P\}) * \phi(\{S_q(y,z)|\ q\in Q\})|\ y\in Y\}) =$$

$$= \phi(\{\phi(\{\phi(\{R_p(x,y)|\ p\in P\}) * S_q(y,z)|\ q\in Q\})|\ y\in Y\}) =$$

$$= \phi(\{\phi(\{\phi(\{R_p(x,y) * S_q(y,z)|\ p\in P\})|\ q\in Q\})|\ y\in Y\}) =$$

$$= \phi(\{R_p(x,y) * S_q(y,z)|\ p\in P,\ q\in Q,\ y\in Y\}),$$

$$\phi(\{R_pS_q|\ p\in P,\ q\in Q\})(x,z) =$$

$$= \phi(\{\phi(\{R_p(x,y) * S_q(y,z)|\ y\in Y\})|\ p\in P,\ q\in Q\}) =$$

$$= \phi(\{R_p(x,y) * S_q(y,z)|\ p\in P,\ q\in Q,\ y\in Y\}),$$

which proves (9). A special case of (9) has the form

$$\bar{R}^2 = \phi(\{R^n|\ n\in \mathbf{N}\})\phi(\{R^k|\ k\in \mathbf{N}\}) =$$

$$= \phi(\{R^{n+k}|\ n\in \mathbf{N},\ k\in \mathbf{N}\}) = \phi(\{RR^m|\ m\in \mathbf{N}\}) =$$

$$= R\phi(\{R^m|\ m\in \mathbf{N}\}) = R\bar{R} = \bar{R}R,$$

which proves the first part of (10). Now by mathematical induction

$$(\bar{R})^n = R^{n-1}\bar{R} \quad \text{for} \quad n \in \mathbf{N},\ n \geq 2$$

and therefore

$$(\bar{\bar{R}}) = \phi(\{(\bar{R})^n|\ n \in \mathbf{N}\}) =$$

$$= \phi(\{\bar{R}\} \cup \{R^{n-1}\bar{R}|\ n \geq 2\}) =$$

$$= \phi(\{\phi(\{R^{n-1}R^k|\ k \in \mathbf{N}\})|\ n \in \mathbf{N}\}) =$$

$$= \phi(\{R^{n+k-1}|\ n,\ k \in \mathbf{N}\}) =$$

$$= \phi(\{R^m|\ m \in \mathbf{N}\}) = \bar{R}.$$

Thus (10) is proven. From Hypothesis 3

$$R = \phi(\{R\}) \leq \phi(\{R^n|\ n \in \mathbf{N}\}) = \bar{R}$$

and using the proof of (10) we also get

$$(\bar{R})^n = \phi(\{R^k|\ k \geq n\}) \leq \phi(\{R^k|\ k \in \mathbf{N}\}) = \bar{R},$$

which proves inequalities (11).

If additionally

(13) $\quad a * 1 = 1 * a = a, \quad a * 0 = 0 * a = 0 \quad$ for $a \in L$,

then from (5) we get

$$(I_X R)(x,z) = \phi(\{I_X(x,y) * R(y,z) \mid y \in Y\}) =$$
$$= \phi(\{1 * R(x,z), \}) = \phi(\{1, 0\}) * R(x,z) = R(x,z),$$
$$(0_{X\times X} R)(x,z) = \phi(\{0 * R(y,z) \mid y \in Y\}) = \phi(\{0\}) = 0$$

for $R \in L(X,X)$, $x,z \in X$. Thus

(14) $\quad I_X R = R I_X = R, \quad 0_{X\times X} R = R 0_{X\times X} = 0_{X\times X} \quad$ for $R \in L(X,X)$.

Therefore $L(X,X)$ forms a semigroup with identity I_X and null $0_{X\times Y}$.

Now we consider connections between fuzzy relations and fuzzy sets.

DEFINITION 3 *). The *image* of fuzzy set $A \in L(X,Y)$ by fuzzy relation $R \in L(X,Y)$ is a fuzzy set $R(A) \in L(Y)$ such that

(15) $\quad R(A)(y) = \phi(\{A(x) * R(x,y) \mid x \in X\}) \quad$ for $y \in Y$.

The *inverse image* of fuzzy set $B \in L(Y)$ under fuzzy relation R is its image $R^{-1}(B) \in L(X)$ by the converse relation R^{-1}, i.e.

(16) $\quad R^{-1}(B)(x) = \phi(\{B(y) * R(x,y) \mid y \in Y\}) \quad$ for $x \in X$.

Special cases of the image and inverse image are *projections* of fuzzy relation $R \in L(X,Y)$ on X or on Y:

(17) $\quad {}_X R = R^{-1}(1_Y), \quad R_Y = R(1_X)$.

The image (15) and inverse image (16) can be considered as boundary cases of composition (5). It suffice to identify X with $\{1\} \times X$ or $L(X)$ with $L(\{1\}, X)$.

LEMMA 1. If $A \in L(X)$, $B \in L(Y)$ and $R \in L(X,Y)$, then

(18) $\quad R(A) = AR, \quad R^{-1}(B) = BR^{-1}$

with interpretation of A and B as boundary relations.

*) Cf. Erceg [4].

P r o o f. $AR \in L(\{1\}, Y) = L(Y)$ directly from Definition 2 and

$$(AR)(1,y) = \emptyset(\{A(1,x) * R(x,y) \mid x \in X\}) =$$

$$= \emptyset(\{A(x) * R(x,y) \mid x \in X\}) = R(A)(y) \quad \text{for} \quad y \in Y.$$

Similarly we get the second equality in (17).

Using the above Lemma the properties of images can be obtained from the properties of composition (5). Thus as a consequence of Theorem 1 we get

THEOREM 2. a) Under Hypothesis 1

(19) $(RS)(A) = S(R(A)) \quad \text{for} \quad R \in L(X,Y), \; S \in L(Y,Z), \; A \in L(X),$

(20) $R^{n+1}(A) = R^n(R(A)), \quad (\bar{R})^{n+1}(A) = \bar{R}(R^n(A))$

for $R \in L(X,X)$, $A \in L(X)$, $n \in \mathbf{N}$,

(21) $\emptyset(\{R_p \mid p \in P\})(\emptyset(\{A_q \mid q \in Q\})) = \emptyset(\{R_p(A_q) \mid p \in P, \; q \in Q\})$

for $R_p \in L(X,Y)$, $A_q \in L(X)$, $p \in P$, $q \in Q$.

b) Under Hypotheses 1 - 3

(22) $I_X(A) = A, \quad R(0_X) = 0_Y, \quad 0_{X \times Y}(A) = 0_Y$

for $R \in L(X,Y)$, $A \in L(X)$.

The application of Definition 3 to the characteristic function of a crisp function $f : X \to Y$ leads to the notions of image and inverse image of fuzzy set by a given function *).

LEMMA 2. Let $f : X \to Y$, $A \in L(X)$, $B \in L(Y)$. Putting

(23) $f(A) = 1_f(A), \quad f^{-1}(B) = (1_f)^{-1}(B)$

and assuming Hypotheses 1 - 3 we get

(24) $f(A)(y) = \emptyset(\{A(x) \mid f(x) = y, \; x \in X\}) \quad \text{for } y \in Y,$

(25) $f^{-1}(B)(x) = B(f(x)) \quad \text{for} \quad x \in X,$

(26) $1_f(1_f)^{-1} \geqslant I_X, \quad (1_f)^{-1}1_f \leqslant I_Y.$

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*) Cf. Bellman and Zadeh [1].

P r o o f. Let $y \in Y$. Then

$$f(A)(y) = \emptyset(\{A(x) * 1_f(x,y) \mid x \in X\}) =$$

$$= \emptyset(\{A(x) * 1 \mid f(x) = y,\ x \in X\} \cup \{A(x) * 0 \mid f(x) \neq y,\ x \in X\}) =$$

$$= \emptyset(\{A(x) \mid f(x) = y,\ x \in X\} \cup \{0\})$$

and we get (24). Similarly

$$f^{-1}(B)(x) = \emptyset(\{B(y)' * 1_f(x,y) \mid x \in X\})$$

$$= \emptyset(\{B(y) * 1 \mid y = f(x)\} \cup \{B(y) * 0 \mid y \neq f(x),\ y \in Y\}) =$$

$$= \emptyset(\{B(f(x)),\ 0\}) \quad \text{for} \quad x \in X,$$

which gives (25).

Using (5) we see that

$$[1_f(1_f)^{-1}](x.z) = \begin{cases} \emptyset(\{0,\ 1\}) = 1, & \text{if } f(x) = f(z) \\ \emptyset(\{0\}) = 0, & \text{otherwise} \end{cases},\ x,\ z \in X,$$

$$[(1_f)^{-1}1_f](y,z) = \begin{cases} \emptyset(\{0,\ 1\}) = 1 & \text{for } y = z \in f(X) \\ \emptyset(\{0\}) = 0, & \text{otherwise} \end{cases},\ y,\ z \in Y,$$

which proves (26).

Using this Lemma and Theorem 1 we obtain

THEOREM 3. Let $f : X \to Y$. Under Hypotheses 1 - 3

(27) $(1_f 1_h(A) = h(f(A))$ for $h : Y \to Z,\ A \in L(X)$,

(28) $f^{-1}(f(A)) \geq A,\ \ f(f^{-1}(B)) \leq B$ for $A \in L(X),\ B \in L(Y)$,

(29) $f(\{\emptyset(A_p) \mid p \in P\}) = \emptyset(\{f(A_p) \mid p \in P\})$ for $A_p \in L(X),\ p \in P$,

(30) $f^{-1}(\{\emptyset(B_q) \mid q \in Q\}) = \emptyset(\{f^{-1}(B_q) \mid q \in Q\})$

for $B_q \in L(Y),\ q \in Q$,

(31) $f(0_X) = 0_Y,\ \ f^{-1}(0_Y) = 0_X$.

Another kind of fuzzy relation can be obtained by the Cartesian product of fuzzy sets:

(32) $(A \times B)(x,y) = A(x) * B(y)$ for $A \in L(X),\ B \in L(Y),\ x \in X, y \in Y$.

This operation is strictly connected with images (15), (16) and projections (17).

THEOREM 4. Let $A \in L(X)$, $B, C \in L(Y)$, $D \in L(Z)$, $R \in L(X,Y)$. Assume Hypotheses 1 – 3. If operation $*$ is isotone, then

(33) $A \times 0_Y = 0_X \times B = 0_{X \times Y}$,

(34) ${}_X(A \times B) \leq A, \quad (A \times B)_Y \leq B$,

(35) ${}_X(A \times 1_Y) = A, \quad (1_X \times B)_Y = B$,

(36) $(A \times B)(A) \leq B, \quad (A \times B)^{-1}(B) \leq A$,

(37) $(A \times B)(C \times D) \leq A \times D$.

P r o o f. Properties (33) – (37) are direct consequences of the assumptions. For example consider the first equality in (35) and inequality (37):

$${}_X(A \times 1_Y)(x) = \phi(\{A(x) * 1\}) = A(x) * \phi(\{1\}) = A(x) \quad \text{for } x \in X,$$

$$(A \times B)(C \times D)(x,z) =$$
$$= \phi(\{(A(x) * B(y)) * (C(y) * D(z)) \mid y \in Y\}) =$$
$$= \phi(\{A(x) * (B(y) * C(y)) * D(z) \mid y \in Y\}) =$$
$$= A(x) * \phi(\{B(y) * C(y) \mid y \in Y\}) * D(z) \leq$$
$$\leq A(x) * 1 * D(z) = (A \times D)(x,z) \quad \text{for } x \in X,\ z \in Z.$$

A full comparison with crisp relations require considering a fuzzy relation between fuzzy sets.

DEFINITION 4 *). $R \in L(X,Y)$ is a fuzzy relation from $A \in L(X)$ to $B \in L(Y)$ if $R \leq A \times B$. In particular, $R \in L(X,X)$ is a *fuzzy relation on fuzzy set* $A \in L(X)$ if $R \leq A \times A$.

The identity fuzzy relation on fuzzy set $A \in L(X)$ is the relation $I_A \in L(X,X)$ such that

(38) $$I_A(x,y) = \begin{cases} A(x) & \text{for } x = y \\ 0 & \text{for } x \neq y \end{cases} \quad x, y \in X.$$

The above notion of fuzzy relation on fuzzy set is ordinary, uniform or regular under respective assumptions from [2] because of regularity of fuzzy set inclusion. Further properties of relation operations will be considered in comparison with the properties of crisp relations.

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*) Cf. Rosenfeld [6].

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