BOOLEAN LATTICE ,FUZZY LATTICE, AND EXTENSION LATTICE

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ABSTRACT

In this paper we introduced the concept of an extension lattice. We from the point of vien of theory of lattice study a contact and a difference between Boolean lattice, fuzzy lattice and an extension. Both a fuzzy lattice and a Boolean lattice are a bounded, but an extension lattice is unbounded. We proved a fuzzy lattice and proper sublattice of an extension lattice are isomorphic.

Keywords: The extension lattice.

I. PREPARATIVE KNOWLEDGE

First some concepts are listed as following:

A partially ordered set is a set P with a binary relation $\leqslant$, which is reflexive, antisymmetric, and transitive.

Definition 1.1 A lattice is a set L of elements, with two binary operations $\wedge$ and $\vee$ which are idempotent, commutative, and associative and which satisfies the absorption laws.

Definition 1.1' A lattice a partially odered set L of elements, with binary relation $\leqslant$, if it satisfies: For any two elements $\alpha, \beta \in L$, have $\alpha \vee \beta \in L$ and $\alpha \wedge \beta \in L$.

May prove that definition 1.1 is equivalent to definition 1.1'.

A lattice $(L, \vee, \wedge)$ is called distributive if in L the distributive laws hold.

A lattice $(L, \vee, \wedge)$ is called a bounded if in L there is a maximun element 1 and minimum element 0. And for any $a \in L$, $0 \leq a \leq 1$ hond. 0 and 1 are called the boundary of a bounded lattice. A bounded lattice is denoted by $(L, \vee, \wedge, 0, 1)$.

Let P is a partially ordered set . A mapping $N: P \to P$ is called a pseudo-complement which satisfies:

(1) For $a, b \in P$ if $a \leq b$ then $N(a) \geq N(b)$;

(2) For any $a \in P$ then $N(N(a)) = a$.

II. BOOLEAN LATTICE, FUZZY LATTICE

Definition 2.1 In a bounded lattice $(L, \vee, \wedge, 0, 1)$ a element $a \in L$ is called a complement of an element $b \in L$, if $a \wedge b = 0$, $a \vee b = 1$. The complement of a element a means a', i.e. $b = a'$.

Definition 2.2 A Boolean lattice B is a distributive lattice which contains elements 0 and 1 with $0 \leq a \leq 1$ for all a, and in which each element $a \in B$ has a complement a' satisfing $a \wedge a' = 0$ and $a \vee a' = 1$ (i.e. the definition of a complement as definition 2.1).

Theorem 2.1 Let $(B, \vee, \wedge, ', 0, 1)$ is a Boolean lattice, then it satisfies following properties:

(B,1) idempotent laws: $a \wedge a = a \vee a = a$;

(B,2) commutative laws: $a \wedge b = b \wedge a$, $a \vee b = b \vee a$;

(B,3) associative laws: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$;
$a \vee (b \vee c) = (a \vee b) \vee c$;

(B,4) absorption laws: $a \wedge (a \vee b) = a \vee (a \wedge b) = a$;

(B,5) distributive laws: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;

(B,6) involution law: $(a')' = a$;

(B,7) De Morgan laws: $(a \wedge b)' = a' \vee b'$, $(a \vee b)' = a' \wedge b'$;

(B,8) zero-one laws: $a \wedge 0=0$, $a \vee o=a$, $a \wedge 1=a$, $a \vee 1=1$;

(B,9) complementation laws: $a \wedge a' = 0$, $a \vee a' = 1$.

Proposition 2.1 In a Boolean lattice $(B, \vee, \wedge, ', 0, 1)$, let $N(a) = a'$, then N is a pseudo-complement over B.

Definition 2.3 In a bounded lattice $(L, \vee, \wedge, 0, 1)$, a element $b \in L$ is called a complement of a element $a \in L$, if $b = 1-a$. The complement of an element $a \in L$ meaned $\neg a$, i.e. $\neg a = 1-a$.

Definition 2.4 A fuzzy lattice F is a distributive lattice which contains elements 0 and 1 with $0 \leqslant a \leqslant 1$ for all a and in which any $a \in F$ has a complement $\neg a$ satisfing $\neg a=1-a$. (i.e. the definition of a complement as definition 2.3)

Theorem 2.2 Let $(F, \vee, \wedge, \neg, 0, 1)$ is a fuzzy lattice then it satifies following properties:

(F,1) idempotent laws: $a \wedge a = a$, $a \vee a = a$;

(F,2) commutative laws: $a \wedge b = b \wedge a$, $a \vee b = b \vee a$;

(F,3) associative laws: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

$a \vee (b \vee c) = (a \vee b) \vee c$;

(F,4) absorption laws: $a \wedge (a \vee b) = a \vee (a \wedge b) = a$;

(F,5) distributive laws: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;

(F,6) involution law: $\neg(\neg a) = a$;

(F,7) De Morgan laws: $\neg(a \wedge b) = (\neg a) \vee (\neg b)$,

$\neg(a \vee b) = (\neg a) \wedge (\neg b)$;

(F,8) zero-one laws: $a \wedge 0 = 0$, $a \vee 0 = a$, $a \wedge 1 = a$, $a \vee 1 = 1$.
In a fuzzy lattice the complementation laws of a Boolean lattice generally are incorrect.

Proposition 2.2 In a fuzzy lattice $(F, \vee, \wedge, \neg, 0, 1)$ let $N(a) = \neg a$, then N is a pseudo-complement over F.

We take note both a Boolean lattice and a fuzzy lattice are a bounder lattices.

III. AN EXTENSION LATTICE

In (1) and (2) first introduced the definition of an extension subset and the operations between a extension subsets as follwing:

Definition 3.1 So called an extension subset $\tilde{A}$ over an universe of discourse U means that for any $u \in U$ corresponds a number $K_{\tilde{A}}(u) \in (-\infty, +\infty)$ is called the relationship degree of u for $\tilde{A}$ such that a one-to-one mapping:

$$K: \quad U \longrightarrow (-\infty, +\infty)$$
$$u \longmapsto K_{\tilde{A}}(u)$$

is called the relationship function of $\tilde{A}$. The all extension subsets over U write as E(U).

Definition 3.2 Let $\tilde{A}, \tilde{B} \in E(U)$, we introduce the operations as follows:

(1) inclution $\tilde{A} \subseteq \tilde{B}$ iff $\forall u \in U$, $K_{\tilde{A}}(u) \leq K_{\tilde{B}}(u)$;

(2) equality $\tilde{A} = \tilde{B}$ iff $\forall u \in U$, $K_{\tilde{A}}(u) = K_{\tilde{B}}(u)$;

(3) intersection $\tilde{C} = \tilde{A} \cap \tilde{B}$ iff $\forall u \in U$, $K_{\tilde{C}}(u) = \min\{K_{\tilde{A}}(u), K_{\tilde{B}}(u)\}$;

(4) union $\tilde{D} = \tilde{A} \cup \tilde{B}$ iff $\forall u \in U$, $K_{\tilde{D}}(u) = \max\{K_{\tilde{A}}(u), K_{\tilde{B}}(u)\}$;

(5) complement $\tilde{S} = \tilde{A}^c$ iff $\forall u \in U$, $K_{\tilde{S}}(u) = -K_{\tilde{A}}(u)$.

Theorem 3.1 $(E(U), \cup, \cap)$ is a distributive lattice.

Proof of this theorem is elementary.

Definition 3.3 In a lattice $(L, \vee, \wedge)$, an element $b \in L$ is called a complement of a element $a \in L$, if $b = -a$. The complement a means a^c, i.e. $a^c = -a$.

Definition 3.4 An extension lattice E is a distributive lattice, and for any element $a \in E$ of which has a complement a^c satisfing $a^c = -a$. (i.e. the definition of a complement as definition 3.3).

Theorem3.2 Let $(E, \vee, \wedge, c)$ is an extension lattice, then it satisfies following properties:

(E,1) idemmpotent laws $a \wedge a = a \vee a = a$;

(E,2) commutative laws $a \wedge b = b \wedge a$, $a \vee b = b \vee a$;

(E,3) associtive laws $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,

$a \vee (b \vee c) = (a \vee b) \vee c$;

(E,4) absorption laws $a \wedge (b \vee a) = a \vee (b \wedge a) = a$;

(E,5) distributive laws $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$;

(E,6) involution law $(a^c)^c = a$;

(E,7) De Morgan laws $(a \wedge b)^c = a^c \vee b^c$, $(a \vee b)^c = a^c \wedge b^c$.

Proposition 3.1 In an extension lattice $(E, \vee, \wedge, c)$, let $N(a) = a^c$, then N is a pseudo-complement over E.

In an extension lattice, the complementation laws and zero-one laws of a Boolean lattice generally are incorrect, and zero-one laws of a fuzzy lattice generally are incorrect, too.

We take note that an extension lattice is not a bounded lattice . Therefore an extension lattice is a now algebraic system.

<u>Theorem 3.3</u> (1) $(E(U), \cup, \cap, c)$ is an extension lattice.

(2) $((-\infty, +\infty), \vee, \wedge, c)$ is an extension lattice, too. (Here the definition of a complement as definition 3.3).

IV. AN EXTENSION SUBLATTICE

<u>Definition 4.1</u> Let S is nonempty subset of an extension lattice $(E, \vee, \wedge, c)$. S is called an extension sublattice, if it satisfies:

(1) $\forall a, b \in S$, $a \vee b \in S$ and $a \wedge b \in S$;

(2) $\forall a \in S$, $a^c \in S$.

<u>Theorem 4.1</u> (1) In the extension lattice $((-\infty, +\infty), \vee, \wedge, c)$, $([-M, M], \vee, \wedge, c)$ is an extension sublattice, and it is a proper extension sublattice of $((-\infty, +\infty), \vee, \wedge, c)$, (Here M is a positive real number).

(2) In an extension lattice $(E(U), \cup, \cap, c)$, an extension sublattice $M(U)$ with is range $[-M, M]$ ($M > 0$ is a positive real number) is an extension sublattice of $E(U)$, and it is a proper extension sublattice of $E(U)$.

In particular, we have follwing

<u>Theorem 4.2</u> Any fuzzy subset $\underset{\sim}{A}$ over an universe of discourse U is isomorphic to some extension subset $\tilde{A} \in M(U)$.

Proof For any $\underset{\sim}{A} \in \mathcal{F}(U)$, let

$$f: \underset{\sim}{A} \to \tilde{A}$$

$$\mu_{\underset{\sim}{A}}(u) \longmapsto 2M \mu_{\underset{\sim}{A}}(u) - M = K_{\tilde{A}}(u)$$

then $\tilde{A} \in M(U)$. (Here M is a positive real number). So that

$$\mu_{\underset{\sim}{A}}(u_1) \vee \mu_{\underset{\sim}{A}}(u_2) \mapsto 2M(\mu_{\underset{\sim}{A}}(u_1) \vee \mu_{\underset{\sim}{A}}(u_2)) - M$$
$$=(2M\mu_{\underset{\sim}{A}}(u_1) - M) \vee (2M\mu_{\underset{\sim}{A}}(u_2) - M)$$
$$=K_{\tilde{A}}(u_1) \vee K_{\tilde{A}}(u_2) \in \tilde{A}$$

Similarly

$$\mu_{A}(u_1) \wedge \mu_{A}(u_2) \mapsto K_{\tilde{A}}(u_1) \wedge K_{\tilde{A}}(u_2) \in \tilde{A}$$

Again from $\mu_{\underset{\sim}{A}}(u) \mapsto 2M\mu_{\underset{\sim}{A}}(u) - M = K_{\tilde{A}}(u)$

have $\mu_{\neg\underset{\sim}{A}}(u) = 1 - \mu_{\underset{\sim}{A}}(u) \mapsto 2M(1 - \mu_{\underset{\sim}{A}}(u)) - M$

$$= 2M - 2M\mu_{\underset{\sim}{A}}(u) - M$$
$$= M - 2M\mu_{\underset{\sim}{A}}(u)$$
$$= -(2M\mu_{\underset{\sim}{A}}(u) - M)$$
$$= -K_{\tilde{A}}(u) = K_{\tilde{A}^c}(u) \in \tilde{A}.$$

Then $(\underset{\sim}{A}, \cup, \cap, \neg) \cong (A, \cup, \cap, c)$

Corollary A fuzzy lattice $\mathscr{F}(U)$ is isomorphic to an extension lattice $M(U)$.

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