ON THE MINIMAL AXIOMATIC SYSTEM OF I-FUZZY STRUCTURE

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1. Introduction

During the last decade the first author made various attempts to axiomatize fuzzy operations starting from a restriction concerning "acceptable" operations - excluding e.g. the widely used min-max representation (see e.g.[1,2]). The clearest establishment of this axiomatic system was named I-fuzzy structure (see [3,4]). Since this time numerous publications appeared which treat fuzzy connectives in a much broader context, esp. basing on the concept of t-norms and t-conorms (see e.g. [5,6]), and these results cover a much more general class of possible fuzzy operations. There are numerous applications of concectives not fulfilling the I-fuzzy axiomatic structure, there exist however several algorithms where a typical representation of I-fuzzy (like e.g. the product and sum minus product pair) is vitally necessary. We mention only one recent paper [7].

So we consider the I-fuzzy axiomatic system not entirely without interest, even at present time. With some study of this system it can be seen that some axioms are redundant. At the same time we also noticed that in this form it is possible to maintain complete symmetry which is, however, quite obvious in the original considerations on the basis of which we established this system. In order to eliminate these problems we show in this paper a minimized and slightly modified version, proving also our statements.

At the end of the Introduction we repeat the I-fuzzy system in the form as it was published in [4].

Definition 1.

$\underline{I} = \langle \otimes, X, \underline{P}, +, *, d, M, \vee, \wedge, \neg \rangle$
is named an I-fuzzy structure if the following axioms are fulfilled:

1. $\underline{P}$ is a bounded ordered set containing its minimum ($\emptyset$) and maximum (1), further on it is closed over the binary operations $+, *$ and d so that ($<$ denotes the ordering in $\underline{P}$):
   - a) $a + b = b + a$
   - b) $(a + b) + c = a + (b + c)$
   - c) $a + 0 = a$
   - d) $(a * b) + (a * c) \geq a * (b + c)$ if $(a * b) + (a * c) \neq 0$ and $a * (b + c) \neq 1$
   - e) $(a + b) * (a + c) \leq a + (b * c)$ if $(a + b) \neq 1$ and $a + (b * c) \neq 0$
   - f) $a + b > a$ if $a \neq 1$ and $b \neq 0$
   - g) $a * b < a$ if $a \neq 0$ and $b \neq 1$
   - h) $a * 0 = 0$
   - i) $d(a,a) = 0$

j) $d(a,b) < d(a,c)$ if $a < b < c$

k) $d(a,b) = d(b,a)$

l) $d(1,0) = 1$

m) $d(1,d(1,a)) = a$

n) $d(1,a + b) = d(1,a) * d(1,b)$

o) $d(a,b) = d(d(1,a),d(1,b))$

p) equation $a + x = b$ $(a \neq 1)$ has maximally one solution for $x \in \underline{P}$

r) equation $a * x = b$ $(a \neq 0)$ has maximally one solution for $x \in \underline{P}$

In the above $a,b,c \in \underline{P}$

2. X is a non-empty set (named the universe).

3. $\textcircled{x}$ is a set having at least two different elements (named the superset of fuzzy sets or statements), $\textcircled{x}$ is closed over the unary operation and the binary operations V and $\wedge$ in the following sense:
There is a one-to-one mapping M, so that $M : \textcircled{x} \to \underline{F}$ where $\underline{F} = \{q \mid q : X \to \underline{P}\}$ (the functions q are named membership functions)
and

a) $\neg A = M^{-1}(d(M(A),M(1)))$

b) $A \vee B = M^{-1}(M(A) + M(B))$

c) $A \wedge B = M^{-1}(M(A) + M(B))$

where $A,B \in \textcircled{x}$

2. The minimal axiomatic system

In this section we reduce and slightly modify the above axioms. Where the proofs are simple, we give them in a brief form. We only go into details if we think that complexity of the proof requires it. In other cases we only give the axioms (in the proper order) from which the proof comes straightforward.

In the following we present 7 Statements: S1 - S7.

S1. "l" can be omitted.

Proof: it comes from "i" and "m" with $a = 1$.

From now on we refer to "S1" instead of "l".

S2. "r" can be omitted.

Proof: Assume indirectly that there exist x,y, P in "p" so that

$$a + x = b \quad (a \neq 1)$$

and

$$a + y = b.$$

Then from "n" we obtain

$$d(1,a) * d(1,x) = d(1,b)$$

and

$$d(1,a) * d(1,y) = d(1,b) \;,\; d(1,a) \neq 0.$$

So there exists an equation in "r" which has two different solutions, and this contradicts to "r".

Similarly, it is easy to prove that if there are such

$x,y \in \underline{P}$ in "r" that

$$a * x = b$$

$$a * y = b \quad a \neq 0,$$

then also an equation according to "p" exists which has two different solutions.

Thus "p" and "r" are equivalent.

Before we go on with our Statements we should like to take some notes. Intuitively, it seems possible to eliminate "e" and "g". In order to do this, we need the following property:

$$j^D \quad d(b,c) < d(a,c) \quad \text{if} \quad a < b < c,$$

which looks very plausible. However, it seems that this cannot be derived from the other axioms. We shall see that if we put "j^D"into the structure instead of "j", "j" remains valid in the new structure, too. Besides, we shall also show later that the "j^D"property is necessary to prove some important statements. So we modify the original structure and use "j^D"instead of "j".
To prove that "j" is valid in the new structure we need the following two statements:

S3. $d(a,0) = a$

Proof: It follows from "o", "S1", "k" and "m" straightforward.

S4. $\forall a \in \underline{P} \ \exists! \ x \in \underline{P}$ that $d(1,x) = a$

Proof: $x = d(1,a)$ from "m".

Now we prove that only one x exists. Let

$$d(1,x) = d(1,y) = z$$

then

$$d(1,z) = d(1,d(1,x)) = d(1,d(1,y))$$

From "m" we obtain $x = y$.

Now we can show that "j" is true in the new structure:

S5. "j" remains valid if it is replaced by "j^D".

Proof:

Case 1. Let $0 < a < b < c < 1$.

From "S4", "j^D", (applied 3 times) and "S1" we obtain

$$0 < d(c,1) < d(b,1) < d(a,1)) < 1.$$

With "j^D"

$$d(d(b,1),d(a,1)) < d(d(c,1),d(a,1)),$$

but from "k" (twice) and "o"

$$d(d(b,1),d(a,1)) = d(a,b)$$

and

$$d(d(c,1),d(a,1)) = d(a,c).$$

So

$$d(a,b) < d(a,c)$$

and this is "j".

Case 2. $0 < a < b < 1$ $(c = 1)$

From "j^D", "S4" and "S1"

$$0 < d(b,1) < d(a,1) < 1$$

from "j^D" we get

$$d(d(b,1),d(a,1)) < d(0,d(a,1))$$

from "k", "o", "k" again and "S3" we get

$$d(a,b) < d(a,1)$$

and this is "j".

Case 3. $0 < b < c < 1$ $(a = 0)$

Applying "S3" and "k" to "b" we get

$$b = d(0,b).$$

Similarly

$$c = d(0,c).$$

However,

$b < c$, so

$d(0,b) < d(0,c)$

and this is "j"

Case 4. $0 < b < 1$ $(a = 0,\ c = 1)$

Like in Case 3.

$b = d(0,b)$

$1 = d(0,1)$

but $b < 1$ so

$d(0,b) < d(0,1)$

and this is "j".

There is no other case.

By "S5" we can state that any statement that is true in the original structure remains valid in the new one, too. In the following, we refer to "S5" instead of "j".

Now we can go on with eliminating some axioms.

S6. "e" can be omitted.

Proof: from "m" and "n" we get

$a * b = d(1,d(1,a) + d(1,b))$

$a * b = d(1,d(1,a) + d(1,c)),$

while

$a * (b + c) = d(1,d(1,a) + d(1,b) * d(1,c))$

comes from "m" and "n" (twice).

Now we can put "d" into the following form

$d(1,d(1,a) + d(1,b) * d(1,c)) < d(1,d(1,a) + d(1,b)) +$
$+ d(1,d(1,a) + d(1,c)).$

Applying "j$^{\flat}$"

$d(1,d(1,d(1,a) + d(1,b)) + d(1,d(1,a) + d(1,c))) <$
$< d(1,d(1,d(1,a) + d(1,b) * d(1,c)))$

thus from "n", "m" and "n" again we get the following:

If we take at first d(1,a), d(1,b) and d(1,c) instead of a,b and c, then (because of "m") we get

$$(a + b) * (a + c) \leq a + (b * c)$$

and this is "e".

We get the necessary conditions for "e" from the conditions referring to "d".

Note: Similarly we could prove that "d" follows from "e" and the other axioms. Thus "d" and "e" are equivalent in this system.

S7. "g" can be omitted.

Proof: it follows from "f" that

$$d(1,a) < d(1,a) + d(1,b)$$

if

$$d(1,a) \neq 1 \Rightarrow a \neq 0$$
$$d(1,b) \neq 0 \Rightarrow b \neq 1$$

Case1. Let $d(1,a) < d(1,a) + d(1,b) < 1$

From "m", "n" and "k"

$$a * b = d(d(1,a) + d(1,b),1).$$

Thus

$$a * b = d(d(1,a) + d(1,b),1) < d(d(1,a),1) = a$$

follows from "j^0", "k" and "m".

Case 2. $d(1,a) < d(1,a) + d(1,b) = 1$

from "m", "n" and "i" we have

$$a * b = d(1,d(1,a) + d(1,b)) = 0$$

however $a > 0$.

Note: It comes from the preceding proof that if

$$x + y = 1$$

then

$$d(1,x) * d(1,y) = 0$$

With this we have finished the reduction and modification of the I-fuzzy axiomatic structure. On the basis of the above we propose the following system (I'-fuzzy):

A(a). $a + b = b + a$

B(b). $(a + b) + c = a + (b + c)$

C(c). $a + 0 = a$

D(d). $(a * b) + (a * c) \geq a * (b + c)$
if $(a * b) + (a * c) \neq 0$ and $a * (b + c) \neq 1$

E(f). $a + b > a$

F(h). $a * 0 = 0$

G(i). $d(a,a) = 0$

H(j^D). $d(b,c) < d(a,c)$ if $a < b < c$

I(k). $d(a,b) = d(b,a)$

J(m). $d(1,d(1,a)) = a$

K(n). $d(1,a + b) = d(1,a) * d(1,b)$

L(o). $d(a,b) = d(d(1,a),d(1,b))$

O(p). Equation $a + x = b$ $(a \neq 1)$ has maximally one solution for $x \in \underline{P}$

3. Some properties of the I'-fuzzy structure

In this section we derive some further properties following from the new structure. The Statements are denoted S8–S18.

S8. $*$ is commutative.

Proof: We get "S8" by applying "J", "K", again "J", "K" and "A" to $(a * b)$ and $(b * a)$, respectively.

S9. $\ast$ is associative.

Proof: We get if by applying "J", "K", "J", "K" and "J" to $(a \ast b) \ast c$ and $a \ast (b \ast c)$, respectively.

S10. $a \ast 1 = a$.

Proof: Applying "J", "C", "K", "J" and "S1" to a we get S10.

S11. $a + 1 = 1$

Proof: Appliyng "S3", "L", "K", "G", "F", "S1", "I" and "S1" again to $a + 1$, we get S11.

S12. $d(1,x+y) < d(1,x)$ if $(x \neq 1,\ y \neq 0)$.

Proof: It comes from "J", "K" and "g" (i.e. "S7").

S13. $d(1,x \ast y) > d(1,x)$ if $(x \neq 0,\ y \neq 1)$.

Proof: $d(1,x) < d(1,x) + d(1,y)$ from "E". Applying "J", "K" and "J" again we get $d(1,x) < d(1,x \ast y)$.

S14. If $a < b < 1$, then $d(1,a \ast b) > d(a,b)$.

Proof: $d(a,b) < d(a,1)$ because of "S5".

We get

$$d(a,b) < d(1,a \ast b)$$

from "I" and "S13".

S15. $d(a,b)$ is unique in the following sense:

(1) To an arbitary $b > a$ there is no $b' > a$ $(b' \neq b)$ so that $d(a,b) = d(a,b')$

(2) To an arbitary $b < a$ there is no $b'' < a$ $(b'' \neq b)$ so that $d(a,b) = d(a,b'')$

Proof:

(1) Case1. $a < b < b'$

$d(a,b) < d(a,b')$ because of "S5".

Case2. $a < b' < b$

$d(a,b') < d(a,b)$ because of "S5".

(2) Case1. $b'' < b < a$

$d(a,b) < d(a,b'')$ because of "H" and "I".

Case2. $b < b'' < a$

$d(a,b'') < d(a,b)$ because of "H" and "I".

<u>S16.</u> If $\exists a^* \in \underline{P}$ that $d(1,a^*) = a^*$, then a^* is unique.

Proof:

Case1. Assume that $a < a^*$ and $a = d(1,a)$.

Then from "H".

$$a^* = d(1,a^*) < d(1,a) = a,$$

which is a contradiction.

Case2. Now let $a^* < a$ and $a = d(1,a)$.

Then from "I" and "H":

$$a = d(1,a) = d(a,1) < d(a^*,1) = d(1,a^*) = a^*,$$

which is again a contradiction.

<u>S17.</u> If $\exists a \in \underline{P}$ that $d(1,a^*) = a^*$, then for arbitrary $a \in \underline{P}$

(1) if $a < a^*$ then $d(1,a) > a^*$

(2) if $a > a^*$ then $d(1,a) < a^*$.

Proof: We prove indirectly, using "S16".

(1) Case1. Let $a < d(1,a) < a^*$.

From "H" and "L":

$d(d(1,a),a^*) < d(a,a^*) =$

$= d(d(1,a),d(1,a^*)) = d(d(1,a),a^*)$,

which is a contradiction.

Case2. $d(1,a) < a < a^*$

From "H", "L" and "J" we obtain

$d(a,a^*) < d(d(1,a),a^*) =$

$d(d(1,d(1,a)),d(1,a^*)) = d(a,a^*)$,

which is again a contradiction.

(2) Case1. $a^* < a < d(1,a)$

From "S5", "L" and "J":

$d(a^*,a) < d(a^*,d(1,a)) =$

$= d(d(1,a^*),d(1,d(1,a)) = d(a^*,a)$,

which is a contradiction.

Case2. $a^* < d(1,a) < a$

From "S5", "J" and "L":

$d(a^*,d(1,a)) < d(a^*,a) =$

$= d(d(1,a^*),d(1,d(1,a)))= d(a^*,d(1,a))$

which is also in contradiction with our assumption.

<u>S18.</u> If $\exists a^* = d(1,a^*)$ and $a_1 < a_2 < a^* < d(1,a_2) < d(1,a_1)$, than

$$d(a_2,d(1,a_2)) < d(a_1,d(1,a_1)).$$

Proof:

$$d(a_2,d(1,a_2)) < d(a_1,d(1,a_2)) < d(a_1,d(1,a_1))$$

because of "S5" and "H".

The above statements help to understand the properties of the connectives and complement (d(1,a)) in the I' system. We should like to stress the specific axiom "O" (formerly "p" and "r") which enables the introduction of inverse logical operations, as they were used e.g. in [1] and several other algorithms. The concept of inverse fuzzy logical operation is not known in the literature.

References

[1] KÓCZY, L.T. - HAJNAL, M.: A New Fuzzy Calculus and Its Application as a Pattern Recignation Technique. In: Modern Trends in Cybernetics and Systems (Eds.: Rose and Bilciu) Vol.2, Springer, Berlin etc., 1977, pp. 103-118.

[2] KÓCZY, L.T.: Interactive σ-Algebras and Fuzzy Objects of Type N. Journ. of Cybernetics, Vol.8. (1978), pp. 273-290.

[3] KÓCZY, L.T.: Vector Valued Fuzzy Sets, BUSEFAL, Été 1980, pp. 41-57.

[4] KÓCZY, L.T.: Vectorial I-fuzzy Sets. In: Approximate Reasoning in Decision Analysis (Eds.: Gupta and Sanchez), North Holland, Amsterdam, etc., 1982, pp. 151-156.

[5] KLEMENT, E.P.: Characterization of Fuzzy Measures Constructed by Means of Triangular Norms, Johannes Kepler Universität, Institut für Mathematik, Institutsbericht Nr. 180, Linz, 1980, pp. 20.

[6] KLEMENT, E.P.: Operations on Fuzzy Sets and Fuzzy Numbers Related to Triangular Norms, 11th IEEE Conf. on Multiple-Valued Logic, Oklahoma City, 1981.

[7] KÓCZY, L.T.: About the Fuzzy Description of Relative Position of Patterns, BUSEFAL 29 (Hiver 1986-87), pp. 58-70 and (different version) Proc. of the 2nd IFSA Congress, Tokyo, 1987., pp. 416-419.